Complex system approach to geospace and climate studies

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Outline of a talk

• Importance of complex system approach
• Phase space reconstruction
• Recurrence plot analysis
• Test for determinism
• Examples
Complex approach

• **Motivation**: Most "real-life" systems are too complicated to be described directly by fundamental laws

• Useful in systems with many degrees of freedom, in the absence of thermodynamic equilibrium (open, dissipative systems), in non-stationary, nonlinear systems

• Morphology is more important than microscopic structure (universality)

• Order emerges spontaneously (**self-organisation**)

**Examples:**
- Chaos, Phase transitions (first and second order), self-organized criticality
Phase space

- All possible states are represented (each possible state of the system corresponding to one unique point in the phase space). For mechanical systems, the phase space usually consists of all possible values of position and momentum variables.

- Questions:
  - How trajectory evolves in the phase space: do trajectories recur (come back) to "same part" of the phase space?
  - Does the recurrence have a period (periodic system)?
  - Do trajectories diverge exponentially (chaotic system) or with a power law (stochastic system)?
  - Are trajectories parallel in the same box of the phase space (deterministic system)?
Phase space reconstruction

• Consider a time series $s(t)$ whose length is $N$

• Embedding dimension is $m$ and $\tau$ is a time delay

• Time delay embedding for $t=1, 2,..., N-(m-1)\tau$ :

$$\mathbf{X}_t = (s_t, s_{t+\tau}, s_{t+2\tau},..., s_{t+(m-1)\tau})$$

$m$-dimensional vector for time series $s$

$\tau$ is the first zero in the autocorrelation function
• For example, for $\tau=2$ and $m=3$:

\[
\vec{X}(t_1) = (s(t_1), s(t_3), s(t_5)) \\
\vec{X}(t_2) = (s(t_2), s(t_4), s(t_6)) \\
\vec{X}(t_3) = (s(t_3), s(t_5), s(t_7))
\]

\[
\vec{X}(t_n) = (s(t_n), s(t_{n+2}), s(t_{n+4}))
\]

• For the Lorenz system:

\[
\{x(t), y(t), z(t)\} \iff \{x(t), x(t + \tau), x(t + 2\tau)\}
\]
(example of Lorenz attractor)

\[
\begin{align*}
\frac{dx}{dt} &= a(y - x) \\
\frac{dy}{dt} &= -xz + cx - y \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\]

- Model of thermal convection in the atmosphere
- Fixed points:
  - \( x^* = y^* = z^* = 0 \)
  - \( x^* = y^* = \pm \sqrt{b(c - 1)}, z^* = c - 1 \)
- \( a=10, b=8/3, \) and \( c=28 \)
Recurrence plots (RPs) visually represent recurrences of the trajectories in the phase space. Suppose we have a trajectory \( \{ \mathbf{x}_i \}_{i=1}^N \) of a system in the phase space.

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= -xz + rx - y \\
\frac{dz}{dt} &= xy - bz
\end{align*}
\]

RP matrix \( R \) consists of zeros and ones; when trajectories recur:

\[
\left| \mathbf{x}_i - \mathbf{x}_j \right| < \varepsilon, \quad R_{i,j}(\varepsilon) = 1.
\]

Here, \( \varepsilon \) is a threshold defined to obtain a fine structure of the RP.

Neighborhood around a point (A): \( L_1 \) norm, (B): \( L_2 \) norm, (C): \( L_\infty \) -norm
A-uniformly distributed white noise
B-superposition of harmonic oscillators
C-logistic map corrupted with linearly increasing term
D-Brownian motion

(A)-segment of the phase space trajectory of the Rössler attractor
(B)-corresponding recurrence plot
RP texture...

- **Homogenous RPs**: stationary systems, where relaxation times are short in comparison with the time spanned by the RP.
- **Periodic and quasiperiodic** systems have RPs with diagonal lines.
- A drift can be seen in non-stationary systems, where RP pales away from the min diagonal.
- **Abrupt changes** in the dynamics as well as extreme events cause white areas or bands in the RP.
Recurrence quantification analysis (RQA)

1. Measures based on vertical lines and recurrent point density

2. Measures based on diagonal lines:

Histogram $P(\varepsilon, l)$ is defined:

$$P(\varepsilon, l) = \sum_{i,j=1}^{N} (1 - R_{i-1,j-1}(\varepsilon))(1 - R_{i+1,j+1}(\varepsilon)) \prod_{k=0}^{l-1} R_{i+k,j+k}(\varepsilon).$$

$$l_{\text{max}} = \max(\{l_i\}_{i=1}^{N_1}) , \quad N_1 = \sum_{l \geq l_{\text{min}}} P(l).$$

$$\langle l^{-1} \rangle = \frac{\sum l^{-1} P(l)}{\sum P(l)}$$
RP for IMF Bz
Storm on 6th of April, 2000
RP for AE index

(two substorm onsets at 5:01 and 21:26)

a) whole day

a) second substorm
Test for determinism

\[
\frac{dx}{dt} = f(x)
\]

\[\Delta \bar{x}(t) = \bar{x}(t + b) - \bar{x}(t)\]

\[b = \text{time spent in the box}\]

\[V_j = \frac{1}{n_j} \sum_{k=1}^{n_j} \frac{\Delta x_{k,j}}{|\Delta x_{k,j}|}\]

\[L_n \equiv \left\langle V_j \right\rangle_{n_j = n}\]

\[n_j = \text{number of passes}\]
Test for determinism (Kaplan and Glass, 1992)

\[
\frac{dx}{dt} = f(x) + w
\]

\[
\Delta \tilde{x}(t) = \bar{x}(t + b) - \bar{x}(t)
\]

\[
L_n \equiv \left\langle V_j \right\rangle_{n_j = n}
\]

\[
v_j = \frac{1}{n_j} \sum_{k=1}^{n_j} \frac{\Delta x_{k,j}}{|\Delta x_{k,j}|}
\]

\[
b = \text{time spent in the box}
\]

Lorenz system
(Phases are randomized in Fourier space)

When randomized $L_n$ vs. $N$ is under the one for the original signal

Process has low-dimensional and nonlinear component

Neither low-dimensional nor nonlinear component
Examples.....
$L_n$ vs. $N$ for
SYM-H data averaged over 10 storms,
for the period 3 days before/after the storm’s main
phase with a resolution of 12 hours

Increased determinism
during the storm
L₆ averaged over 10 storms, for period 3 days before/after the storm’s main phase, with a resolution of 12 hours.
SYM-H* has also increased determinism during storm.

Exclude contribution from magnetopause current:

\[ SYM - H^* = 0.77(SYM - H) - 11.9 \sqrt{P_{\text{dyn}}} \]

\( P_{\text{dyn}} \) is Solar wind (dynamical) pressure.
Determinism in AE index

AE index has low-dimensional, nonlinear component (the same result is obtained for AL, AU and PC index)
Mean $L_n$ over substorms
(database from Frey & Mende, 2002)

AE index is more deterministic during substorms.

During all times

The same is shown for AU and PC (polar cup) index.
a) $B_z$

b) $V$

during substorms

during all times