The *k*-filtering technique applied to Cluster EFW and STAFF data

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Introduction

It is of great interest in wave studies to determine the wave-field energy distribution, P, as a function of frequency, ω, and wave-vector, $\boldsymbol{k}$, for the observed waves.

The ω dependence can easily be found by Fourier transformation of the time series.

The $\boldsymbol{k}$ dependence must be found with other methods because of the limited number of measuring points.

k-filtering is a multi-spacecraft data analysis technique that enables the estimation of $P(\omega, \boldsymbol{k})$.

Introduction

The four Cluster satellites were launched in 2000 and provide multipoint measurements in the Earth's vicinity.

The EFW instruments measure time series of two electric field components.

The STAFF-SC instruments measure time series of the magnetic field fluctuations.

k-filtering

The *k*-filtering technique utilises a filter bank method.

This filter bank is constructed to absorb all signals, except those corresponding to plane waves with a specified frequency and wave vector, which pass untouched.

By going through all frequencies and wave vectors, we get an estimate of the wave-field energy distribution in the signal.

k-filtering – notations

The wave field consists of *L* real quantities:

$$\boldsymbol{A}(\boldsymbol{r},t)=\begin{pmatrix} A_1(\boldsymbol{r},t) \\ A_2(\boldsymbol{r},t) \\ \vdots \\ A_L(\boldsymbol{r},t) \end{pmatrix}$$

The Fourier transformed measurements from *N* spacecraft are put into one vector:

$$\boldsymbol{A}(\omega)=\begin{pmatrix} \boldsymbol{A}(\boldsymbol{r}_1,\omega) \\ \boldsymbol{A}(\boldsymbol{r}_2,\omega) \\ \vdots \\ \boldsymbol{A}(\boldsymbol{r}_N,\omega) \end{pmatrix}$$

A spatial correlation matrix is defined:

$$\boldsymbol{M}(\omega)=\left\langle \boldsymbol{A}(\omega)\boldsymbol{A}^{T*}(\omega)\right\rangle$$

The *k*-filtering equation

Wave-field energy density distribution

$$\boldsymbol{M}(\omega)=\left\langle \boldsymbol{A}(\omega)\boldsymbol{A}^{T*}(\omega)\right\rangle$$

$$\boldsymbol{A}(\omega)=\begin{pmatrix}\boldsymbol{A}(\boldsymbol{r}_1,\omega)\\ \boldsymbol{A}(\boldsymbol{r}_2,\omega)\\ \vdots\\ \boldsymbol{A}(\boldsymbol{r}_N,\omega)\end{pmatrix}$$

$$\boldsymbol{A}(\boldsymbol{r},t)=\begin{pmatrix}A_1(\boldsymbol{r},t)\\ A_2(\boldsymbol{r},t)\\ \vdots\\ A_L(\boldsymbol{r},t)\end{pmatrix}$$

Spatial correlation matrix

Data from *N* different spacecraft

Wave-field of *L* real quantities

$$P(\omega,\boldsymbol{k})=\mathrm{Tr}\left\{\boldsymbol{C}(\omega,\boldsymbol{k})\left(\boldsymbol{C}^{T*}(\omega,\boldsymbol{k})\boldsymbol{H}^{T*}(\boldsymbol{k})\boldsymbol{M}^{-1}(\omega)\boldsymbol{H}(\boldsymbol{k})\boldsymbol{C}(\omega,\boldsymbol{k})\right)^{-1}\boldsymbol{C}^{T*}(\omega,\boldsymbol{k})\right\}$$

Constraining matrix

Matrix to keep track of the spacecraft positions

$$\boldsymbol{H}(\boldsymbol{k})=\begin{pmatrix}\boldsymbol{I}_L e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1}\\ \boldsymbol{I}_L e^{i\boldsymbol{k}\cdot\boldsymbol{r}_2}\\ \vdots\\ \boldsymbol{I}_L e^{i\boldsymbol{k}\cdot\boldsymbol{r}_N}\end{pmatrix}$$

L×*L* unit matrix

Simple 1D example

Two satellites (at x_1 and x_2) measuring one field quantity $\Phi(x,t)$.

The wave field is given by:

$$\phi(x,\omega)=\phi_0(\omega)\mathrm{e}^{ik_0 x}$$

The spatial correlation matrix is then:

$$\boldsymbol{M}(\omega)=\left|\phi_0(\omega)\right|^2\begin{pmatrix} 1 & \mathrm{e}^{ik_0(x_1-x_2)} \\ \mathrm{e}^{-ik_0(x_1-x_2)} & 1 \end{pmatrix}$$

This is not invertible, we must thus add some incoherent noise:

$$\boldsymbol{M}(\omega)=\left|\phi_0(\omega)\right|^2\begin{pmatrix} 1+\varepsilon & \mathrm{e}^{ik(x_1-x_2)} \\ \mathrm{e}^{-ik(x_1-x_2)} & 1+\varepsilon \end{pmatrix}$$

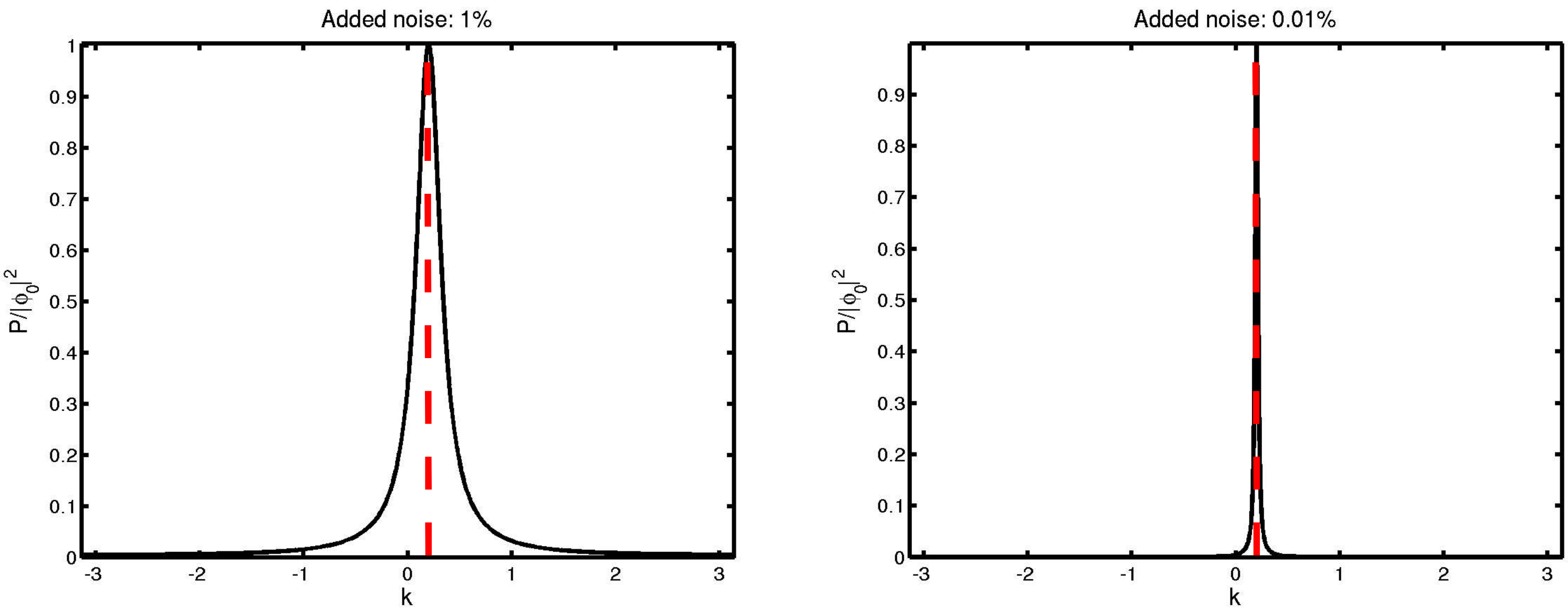
The ***H***-matrix in this case is:

$$\boldsymbol{H}(k)=\begin{pmatrix} \mathrm{e}^{ikx_1} \\ \mathrm{e}^{ikx_2} \end{pmatrix}$$

Simple 1D example

This gives (after some managable algebra):

$$P(\omega,k)=\left|\phi_0(\omega)\right|^2 \frac{\varepsilon(2+\varepsilon)}{2\left(1+\varepsilon-\cos\left[\left(k-k_0\right)\left(x_1-x_2\right)\right]\right)}$$

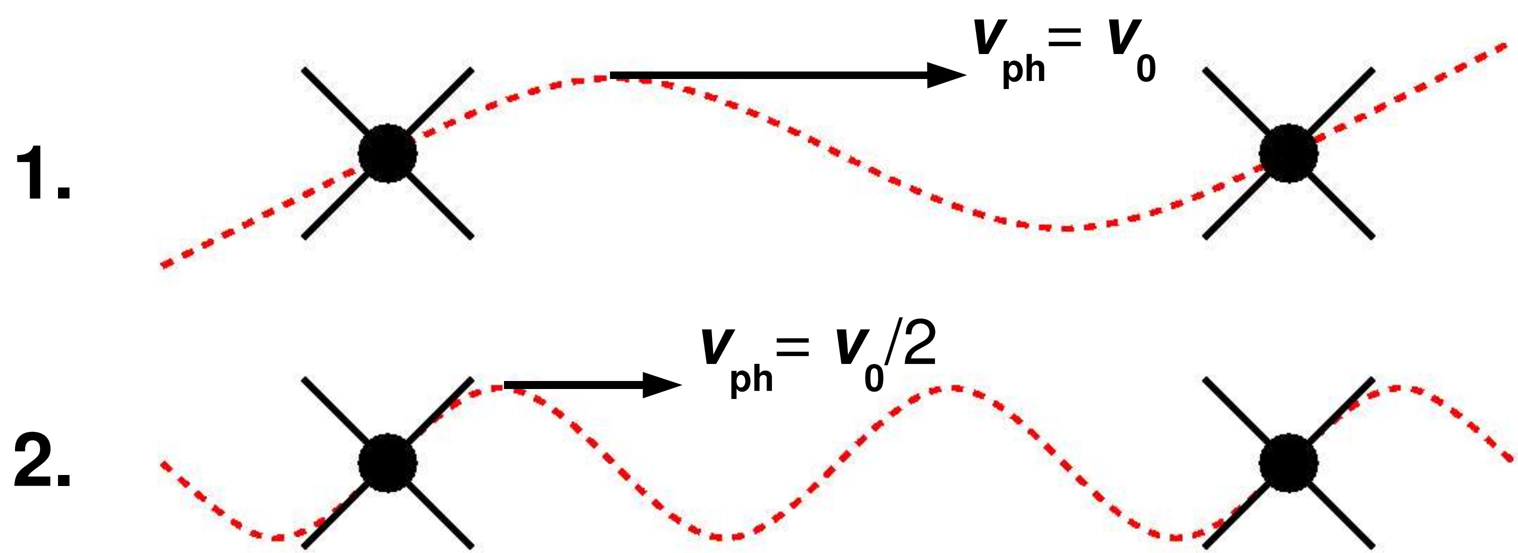


$k_0 = 0.2;\ x_1 - x_2 = 1$

Note that the result is periodic in *k*!

Spatial aliasing

Two spacecraft cannot distinguish between these two situations:



This gives rise to the so called “spatial aliasing”: an ambiguity in the determination of ***k***-vectors.

Spatial aliasing in 3D

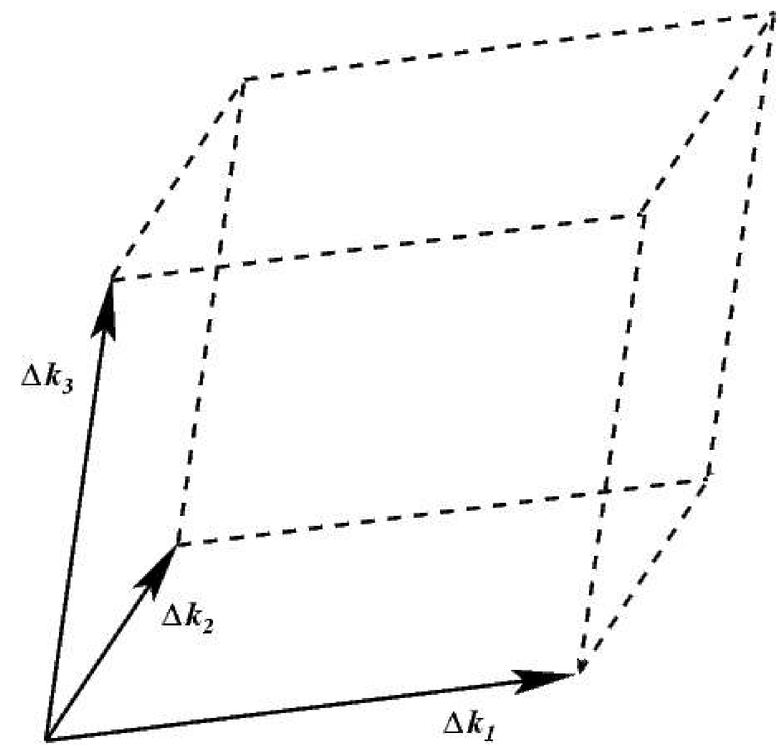
We know that $\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \mathrm{e}^{i(\boldsymbol{k}+\Delta\boldsymbol{k})\cdot\boldsymbol{r}}$, when $\Delta\boldsymbol{k}\cdot\boldsymbol{r} = 2\pi n$ and *n* is an integer.

This leads to $\Delta\boldsymbol{k} = n_1\,\Delta\boldsymbol{k}_1 + n_2\,\Delta\boldsymbol{k}_2 + n_3\,\Delta\boldsymbol{k}_3$, with

$$\Delta\boldsymbol{k}_1 = 2\,\pi(\boldsymbol{r}_2 - \boldsymbol{r}_4)\times(\boldsymbol{r}_3 - \boldsymbol{r}_4)/V$$
$$\Delta\boldsymbol{k}_2 = 2\,\pi(\boldsymbol{r}_3 - \boldsymbol{r}_4)\times(\boldsymbol{r}_1 - \boldsymbol{r}_4)/V$$
$$\Delta\boldsymbol{k}_3 = 2\,\pi(\boldsymbol{r}_1 - \boldsymbol{r}_4)\times(\boldsymbol{r}_2 - \boldsymbol{r}_4)/V$$
$$V = (\boldsymbol{r}_1 - \boldsymbol{r}_4)\cdot\left[(\boldsymbol{r}_2 - \boldsymbol{r}_4)\times(\boldsymbol{r}_3 - \boldsymbol{r}_4)\right]$$

when we have four satellites.

We get a parallelepiped in *k*-space that describes the aliasing properties.



Another simple 1D example

Two satellites (at x_1 and x_2) measuring two monochromatic waves, $\Phi_1(x,t)$ and $\Phi_2(x,t)$, with the same frequency, ω, but different wavenumbers, k_1 and k_2. The two waves are assumed to not be phase coherent.

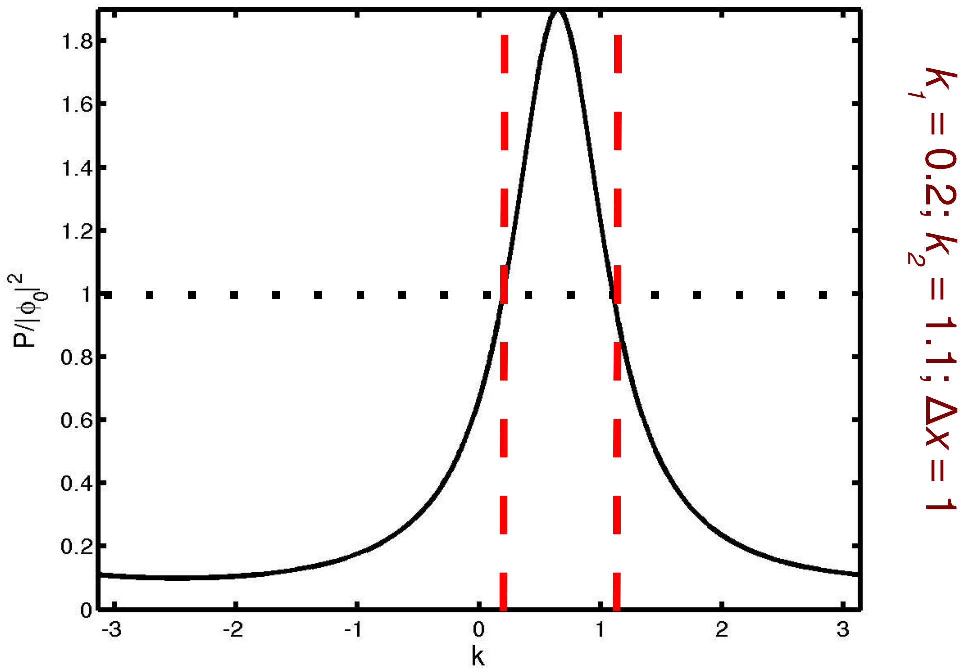
Some algebra gives:

$$P(\omega,k)=\frac{\left|\phi_1(\omega)\right|^2\left|\phi_2(\omega)\right|^2\left(1-\cos\left[(k_1-k_2)(x_1-x_2)\right]\right)}{\left|\phi_1(\omega)\right|^2\left(1-\cos\left[\left(k_1-k_0(\omega)\right)\left(x_1-x_2\right)\right]\right)+\left|\phi_2(\omega)\right|^2\left(1-\cos\left[\left(k_2-k_0(\omega)\right)\left(x_1-x_2\right)\right]\right)}$$

No noise needs to be added this time, but...

Another simple 1D example

The final result has only one peak.



There are two different ways to resolve the two wavenumbers.

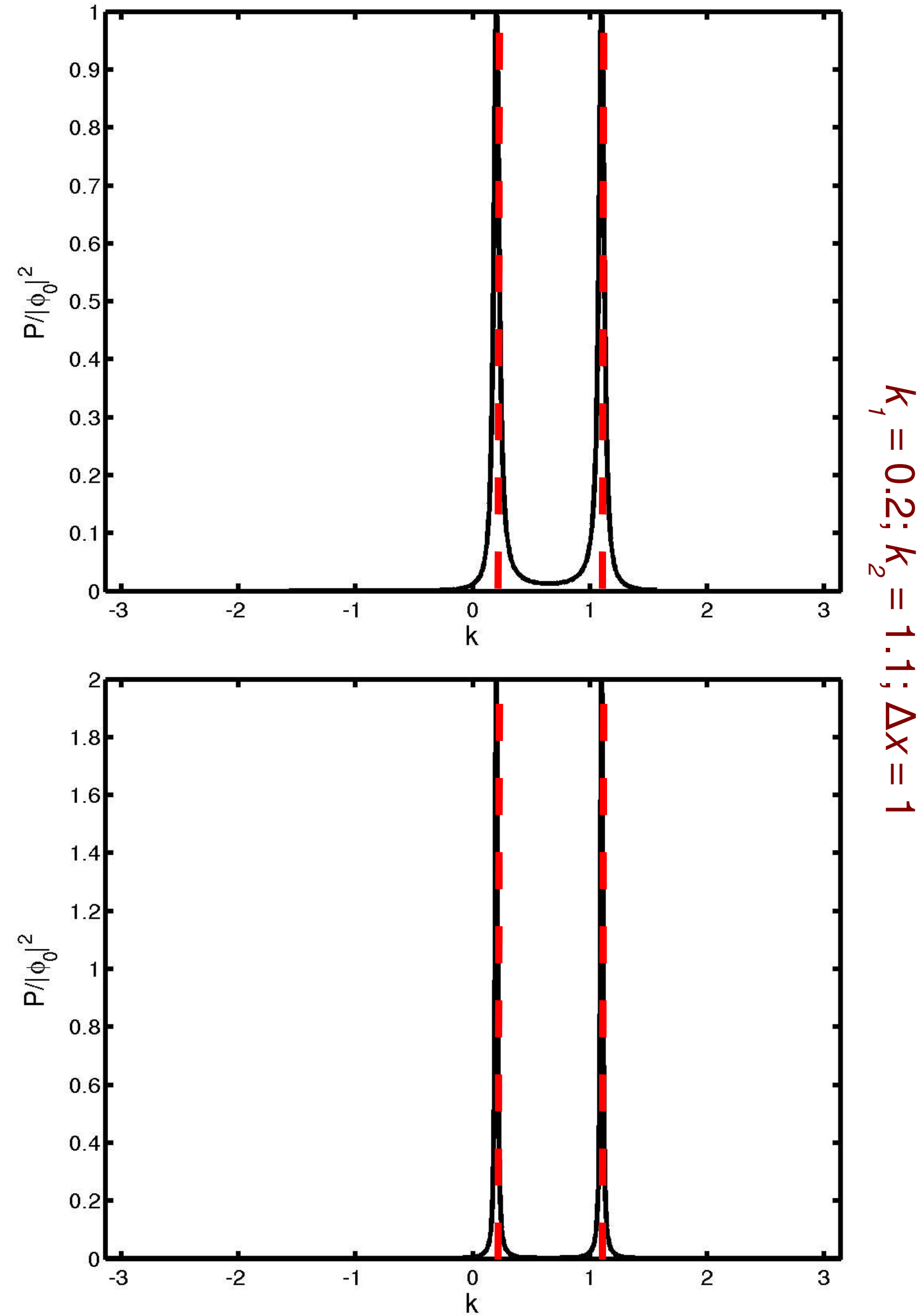
Another simple 1D example

We can use three satellites. Then we can see both peaks in the energy density distribution:

The more satellites we use, the better resolution we get.

Two satellites are also sufficient, if we look at two or more field quantities:

The more field quantities we use, the better resolution we get.



The constraining matrix

The extra information that we have from physical laws can be included to further enhance the *k*-filtering.

Faraday's law can for example be used for electromagnetic fields.

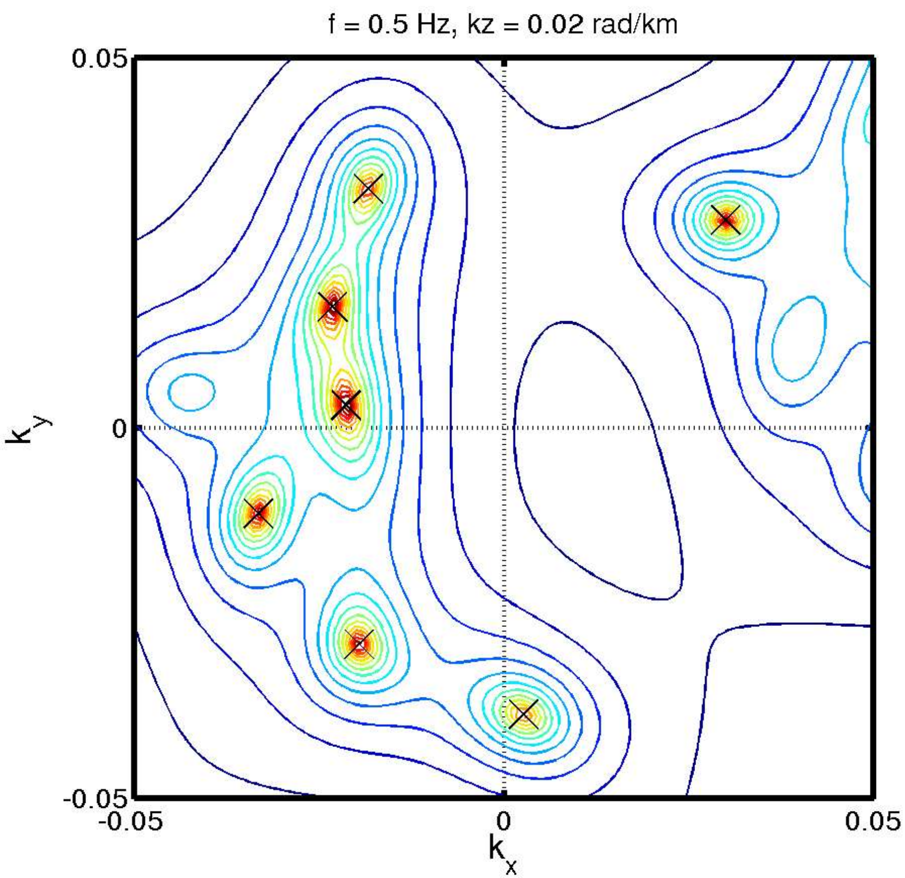
$$\boldsymbol{k}\times\boldsymbol{E}(\omega,\boldsymbol{k})=\omega\boldsymbol{B}(\omega,\boldsymbol{k})$$

This can be used to determine a constraining matrix $\mathbf{C}(\omega,\boldsymbol{k})$:

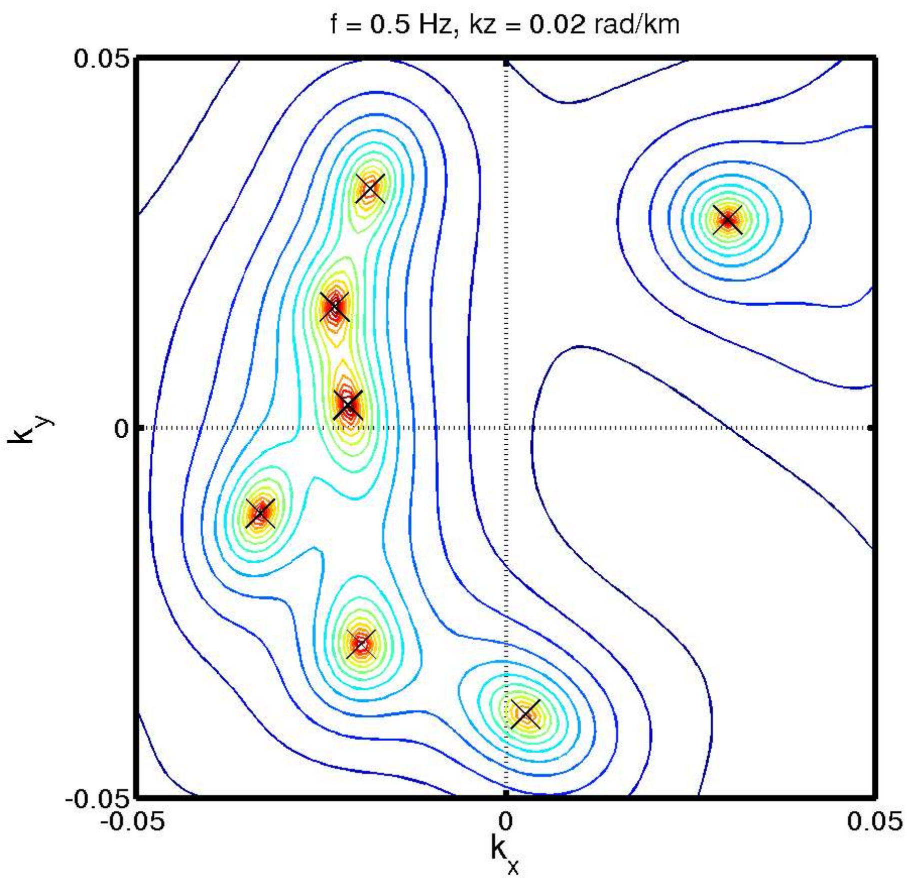
$$\begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ E_z(\omega,\boldsymbol{k}) \\ cB_x(\omega,\boldsymbol{k}) \\ cB_y(\omega,\boldsymbol{k}) \\ cB_z(\omega,\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -ck_z/\omega & ck_y/\omega \\ ck_z/\omega & 0 & -ck_x/\omega \\ -ck_y/\omega & ck_x/\omega & 0 \end{pmatrix} \begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ E_z(\omega,\boldsymbol{k}) \end{pmatrix} = \mathbf{C}(\omega,\boldsymbol{k}) \begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ E_z(\omega,\boldsymbol{k}) \end{pmatrix}$$

Some examples with artificial data

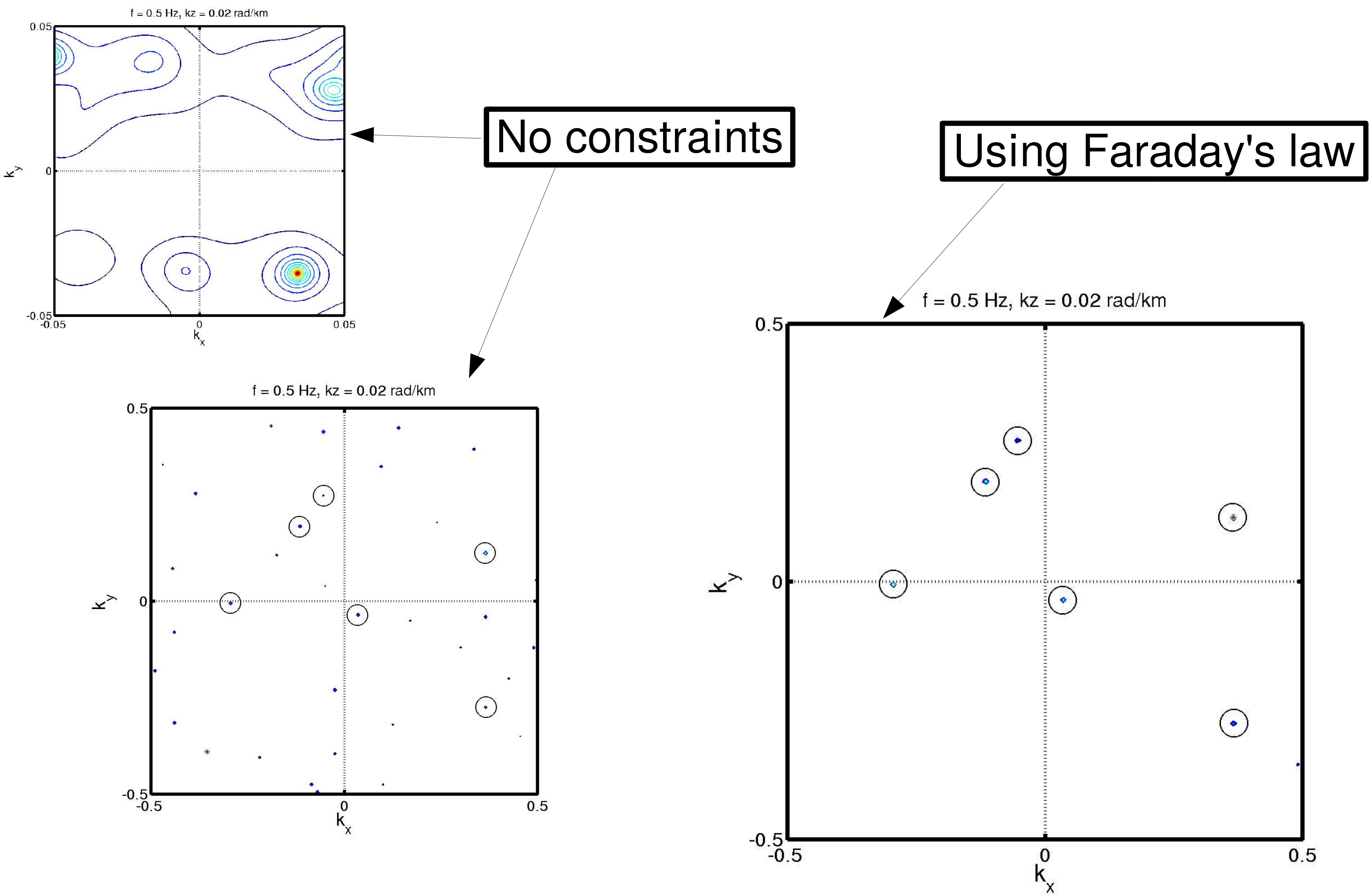
No constraints:



Using Faraday's law:

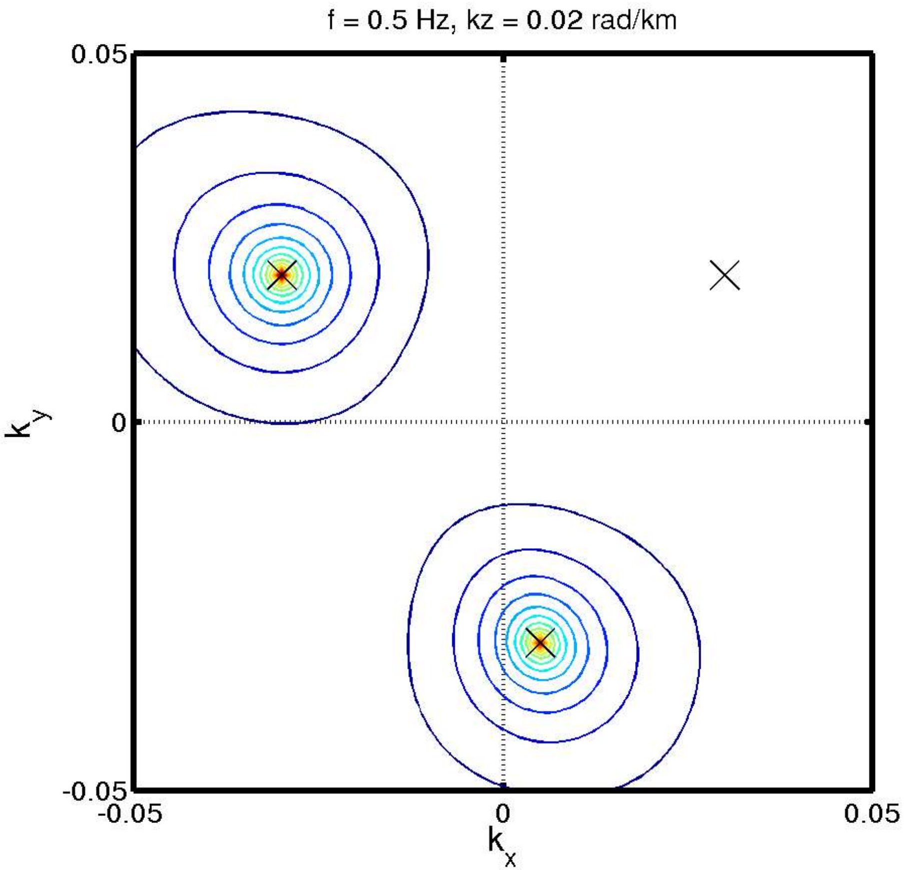


Some examples with artificial data

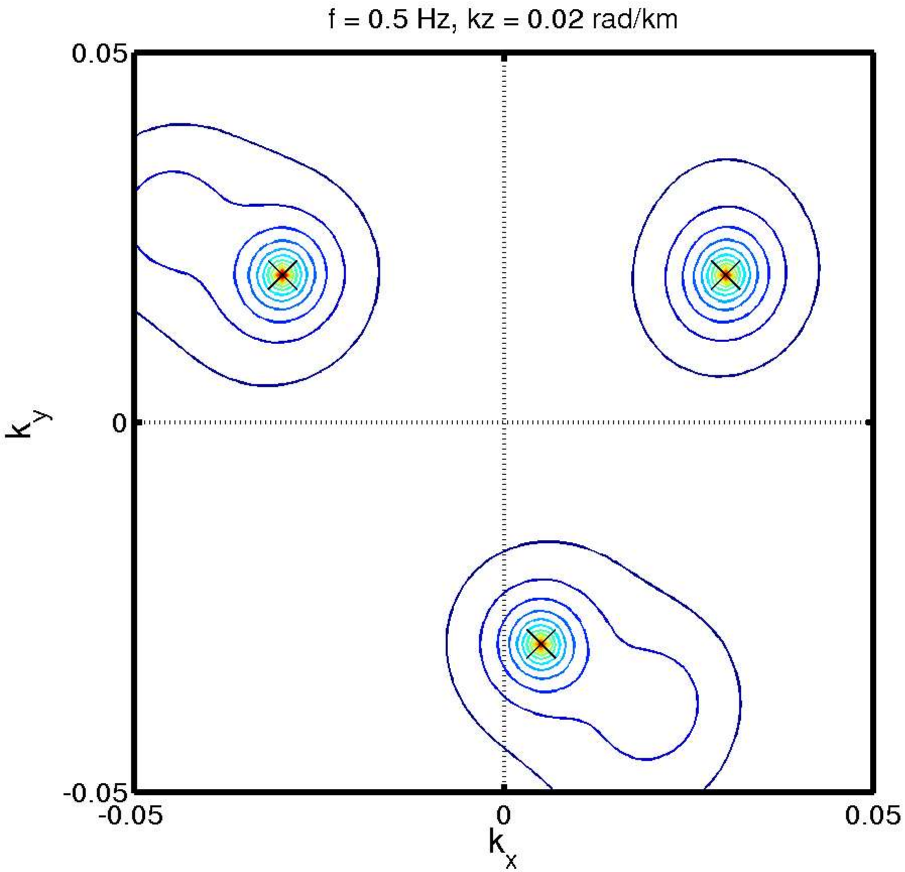


Some examples with artificial data

With only magnetic field:



With magnetic and electric fields:



Problem

The Cluster EFW data has only two components.

We must either reconstruct the missing third component, or find another constraining matrix to use.

We have chosen the second method, where the five measured field-components are parametrised by the three electric field components.

$$\begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ cB_x(\omega,\boldsymbol{k}) \\ cB_y(\omega,\boldsymbol{k}) \\ cB_z(\omega,\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -ck_z/\omega & ck_y/\omega \\ ck_z/\omega & 0 & -ck_x/\omega \\ -ck_y/\omega & ck_x/\omega & 0 \end{pmatrix} \begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ E_z(\omega,\boldsymbol{k}) \end{pmatrix} = \boldsymbol{C}(\omega,\boldsymbol{k}) \begin{pmatrix} E_x(\omega,\boldsymbol{k}) \\ E_y(\omega,\boldsymbol{k}) \\ E_z(\omega,\boldsymbol{k}) \end{pmatrix}$$

Another problem

We use both the electric and magnetic fields to construct the spatial correlation matrix.

Its elements are thus of different orders of magnitude.

We must normalise the elements in order to avoid numerical problems.

Solution

Average over all spacecraft

We use $$R(\omega)=\sqrt{\frac{\langle |E_x(\omega)|^2 + |E_y(\omega)|^2 \rangle}{\langle |B_x(\omega)|^2 + |B_y(\omega)|^2 + |B_z(\omega)|^2 \rangle}}$$

and replace $\boldsymbol{B}(\omega)$ by $R(\omega)\boldsymbol{B}(\omega)$ in the spatial correlation matrix.

We must furthermore replace the constraining matrix $\mathbf{C}(\omega,\boldsymbol{k})$ by $\mathbf{C}(\omega/R(\omega),\boldsymbol{k})$.

Some more problems

There are some technical problems on the EFW instruments:

2001-12-28: Probe failure on spacecraft 1, probe 1
2002-07-29: Probe failure on spacecraft 3, probe 1

This makes data from one or more spacecraft unavailable for use with the *k*-filtering technique.

After September 29, 2003, data from the affected spacecraft can be used for *k*-filtering again.

The *k*-filtering technique can still be used for these cases of non-similar spacecraft, after some modifications of the equations.

Yet another problem

We typically need waves with wave-lengths larger than the spacecraft separation distances, to keep the spatial aliasing under control.

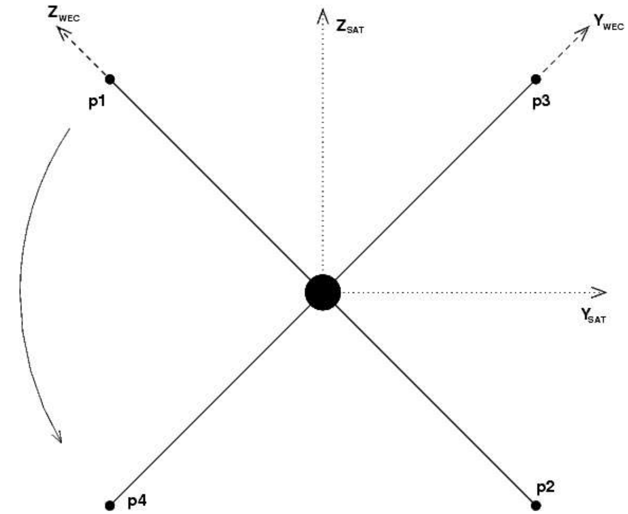
The STAFF-SC magnetic field data we can use has a lowest frequency of ~0.35 Hz.

These two facts means that there is a rather small frequency range around the spacecraft spin frequency (0.25 Hz) and its lowest harmonics where we can use the *k*-filtering with Cluster STAFF and EFW data.

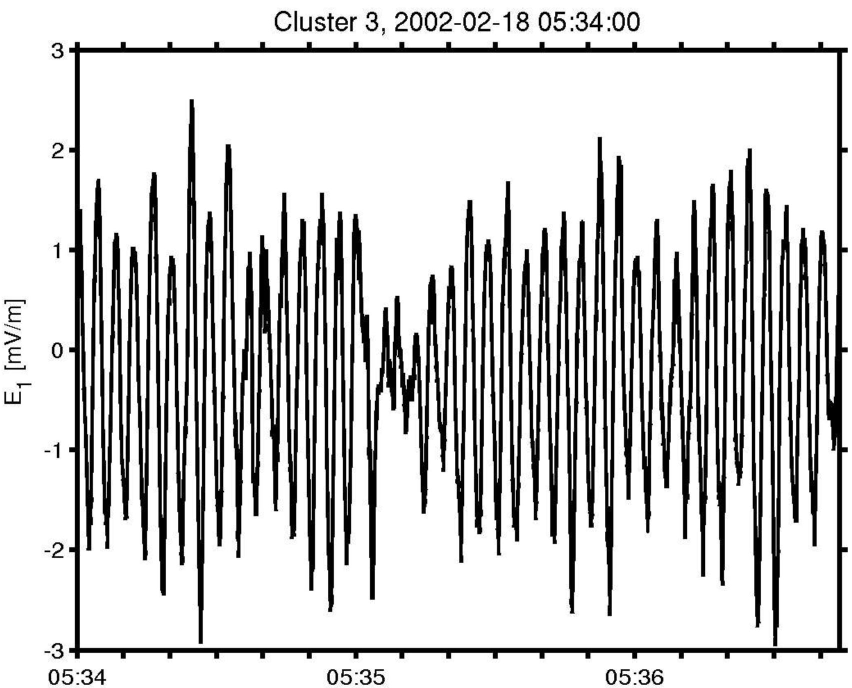
This frequency range is the worst possible one for the electric field measurements, because of asymmetries in the plasma around the spacecraft. The effects of this problem should be at exactly these frequencies.

EFW data cleaning

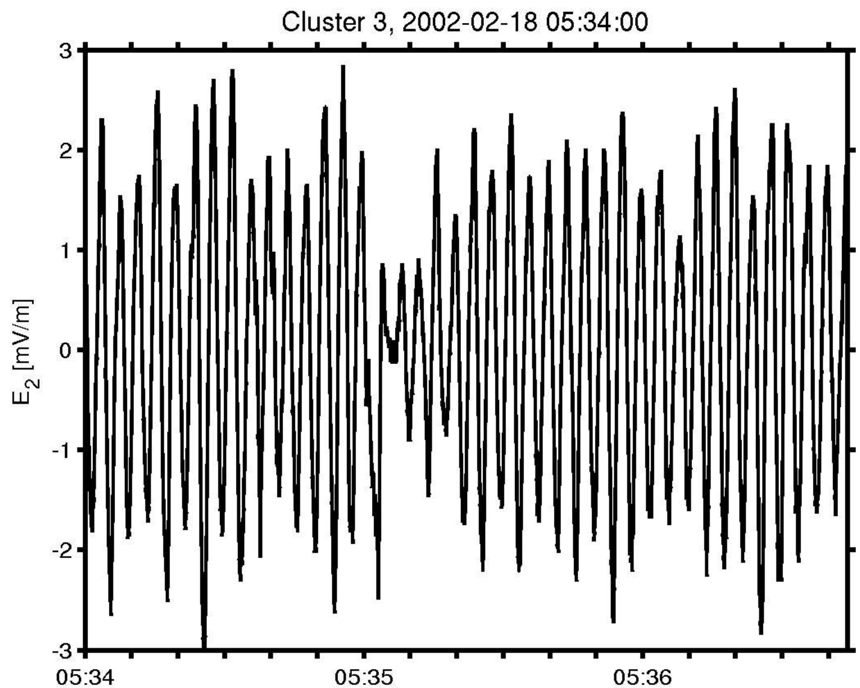
Data from 2002-02-18
05:34:00 - 05:36:44



Electric field between
probes 1 and 2

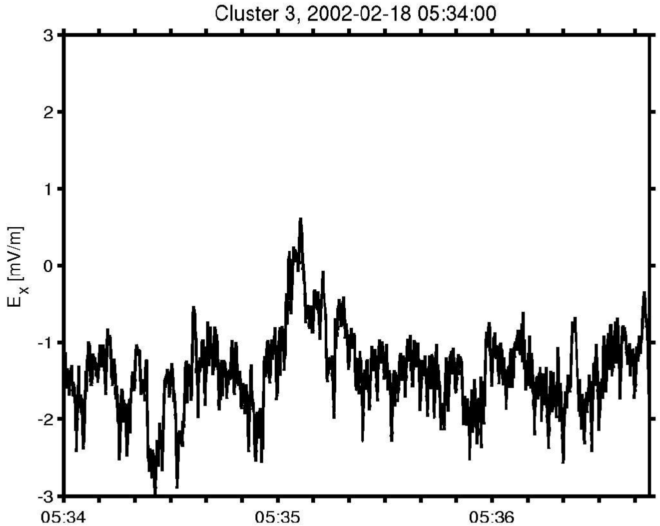


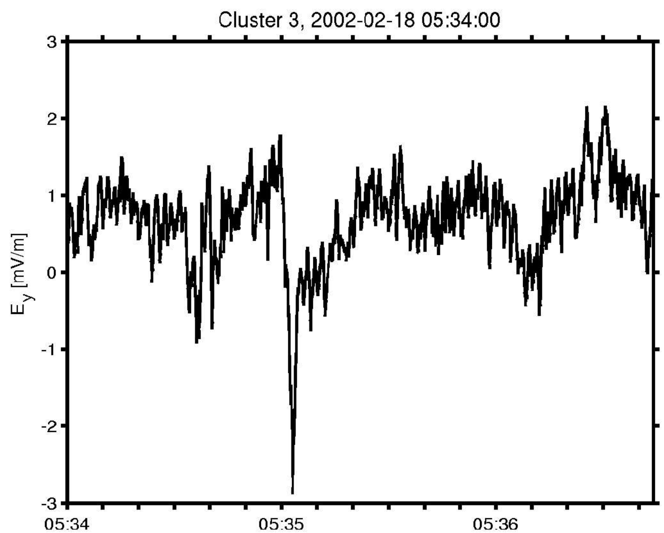
Electric field between
probes 3 and 4



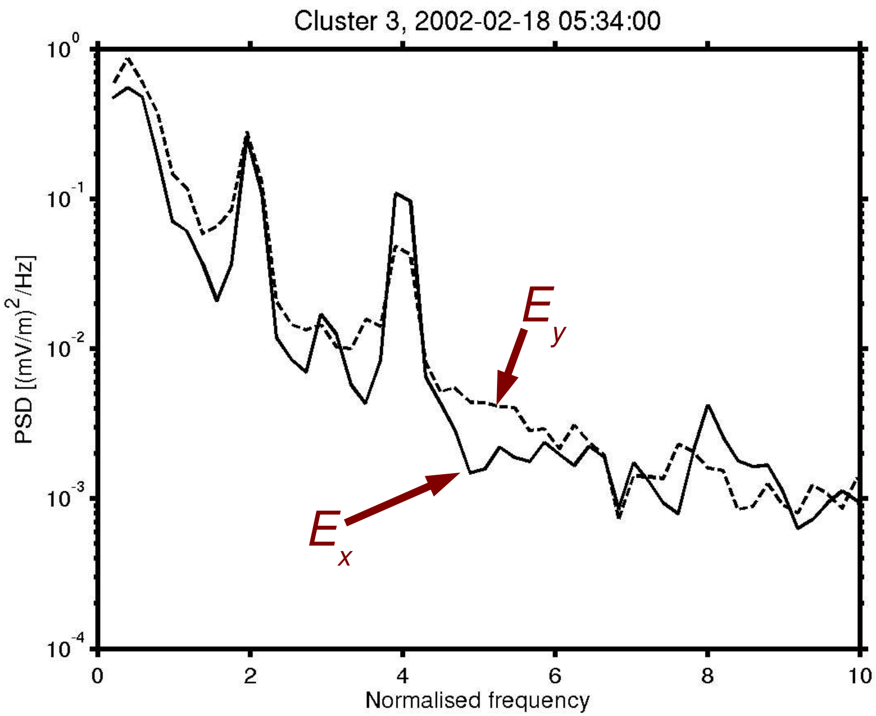
EFW data cleaning

E_x and E_y obtained by direct despinning



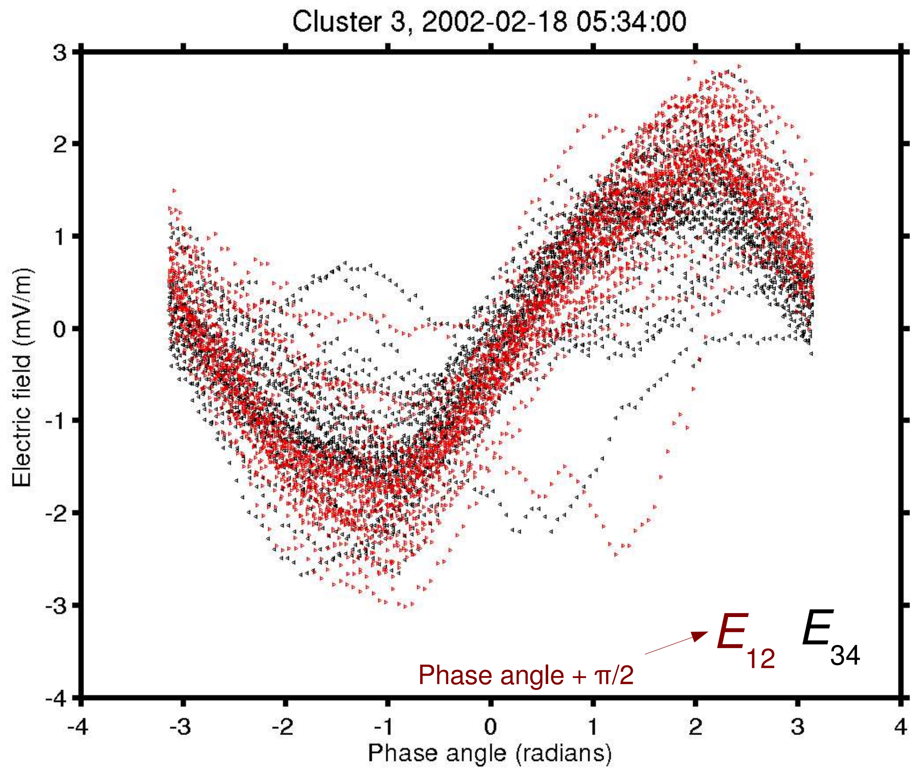


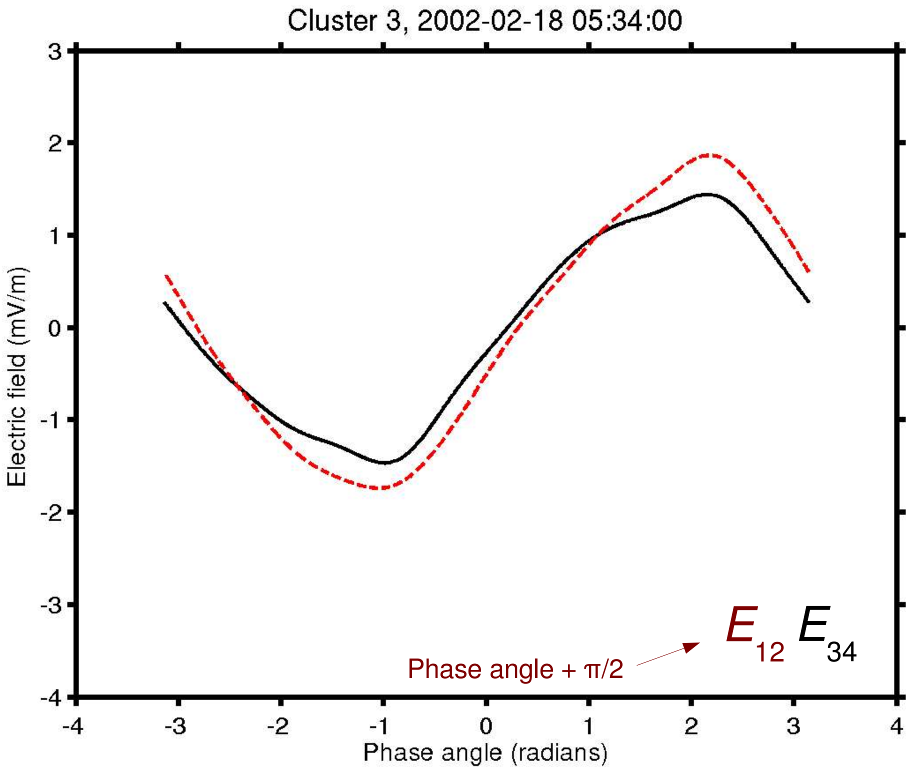
Power spectra for these data



EFW data cleaning

The electric field as a function of phase angle

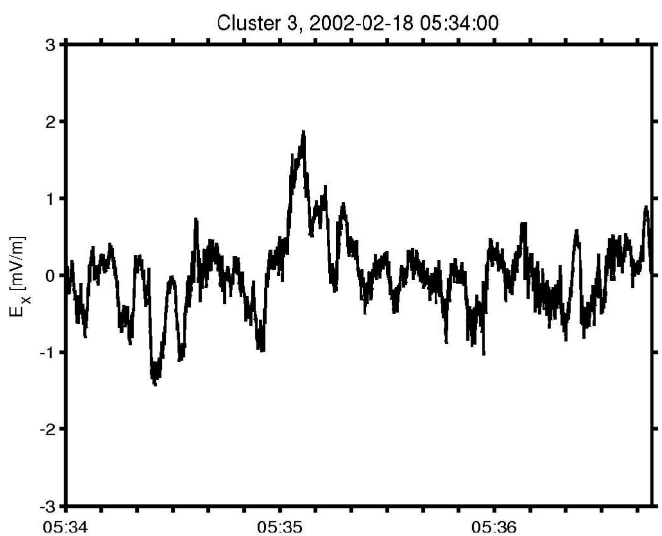


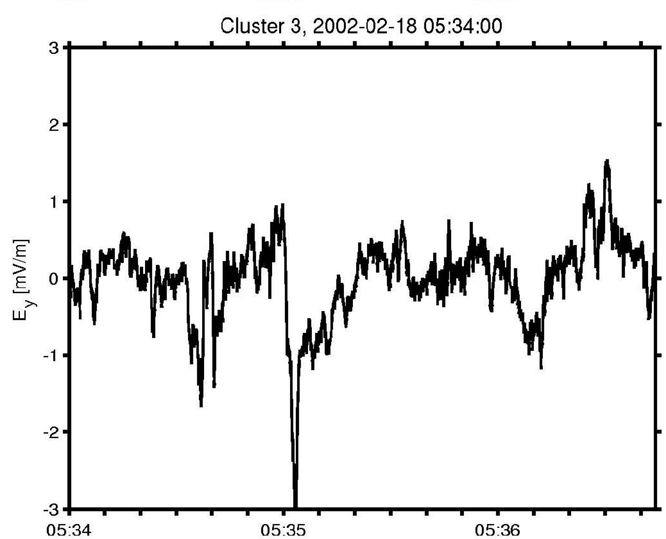


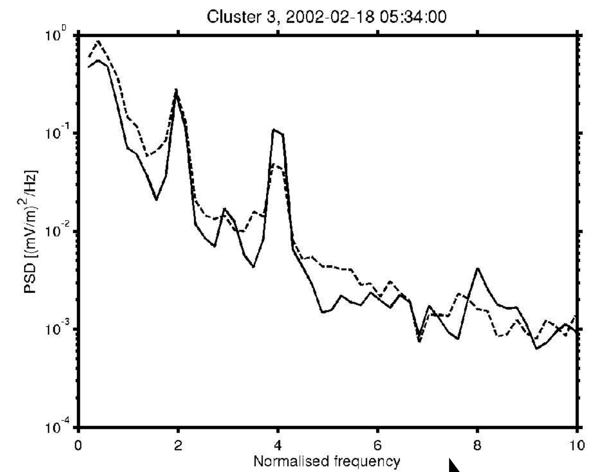
It is convenient to filter the data in this phase-angle domain.

EFW data cleaning

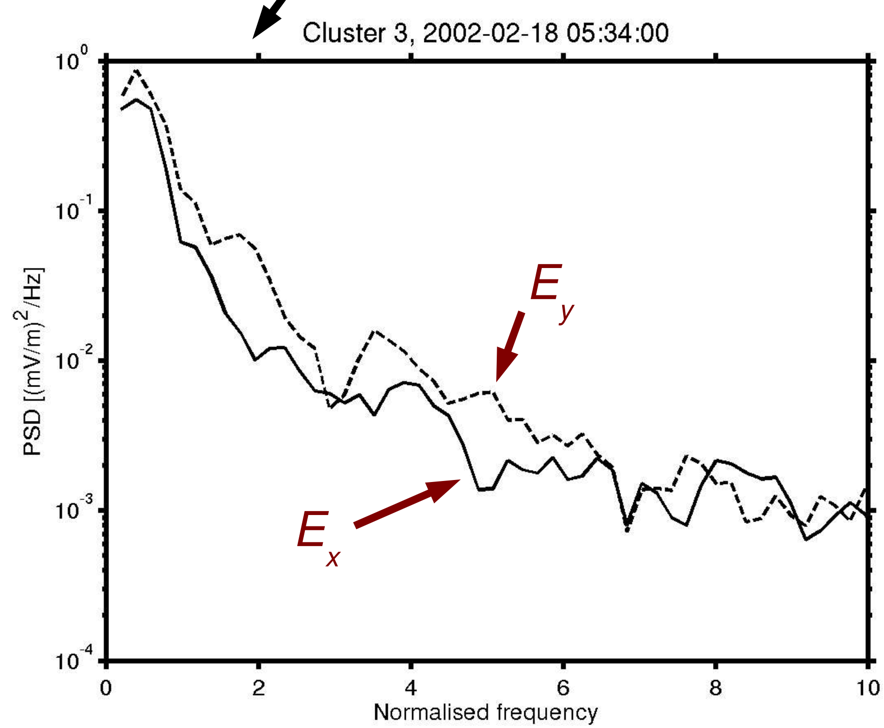
E_x and E_y obtained after removing spin-harmonics







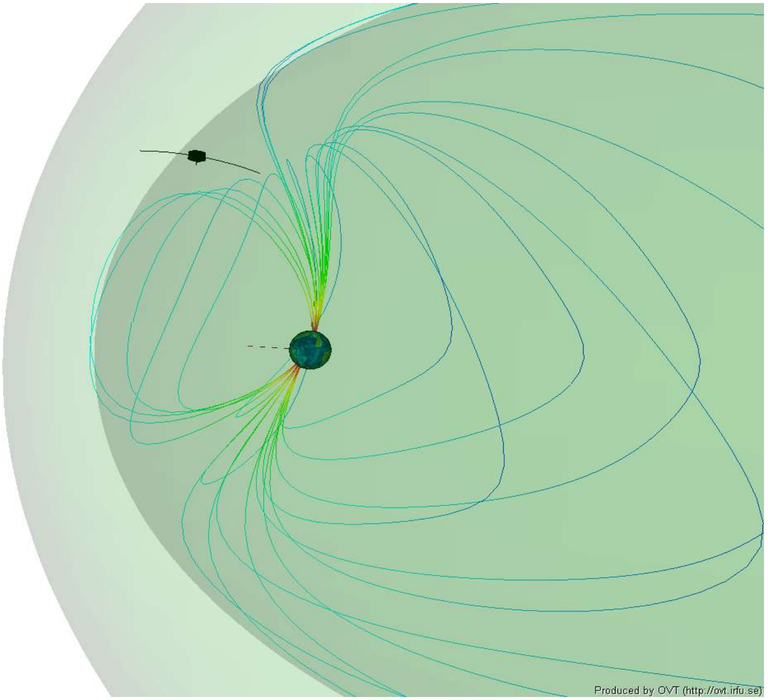
Power spectra obtained before and after this treatment



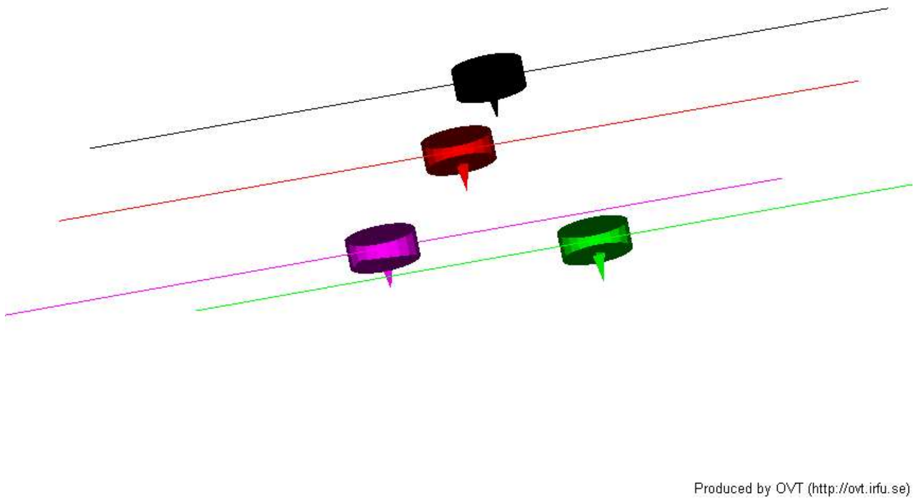
Finally – some *k*-filtering

Example from 2002-02-18 05:34:00 - 05:36:44

Cluster is in the magnetosheath

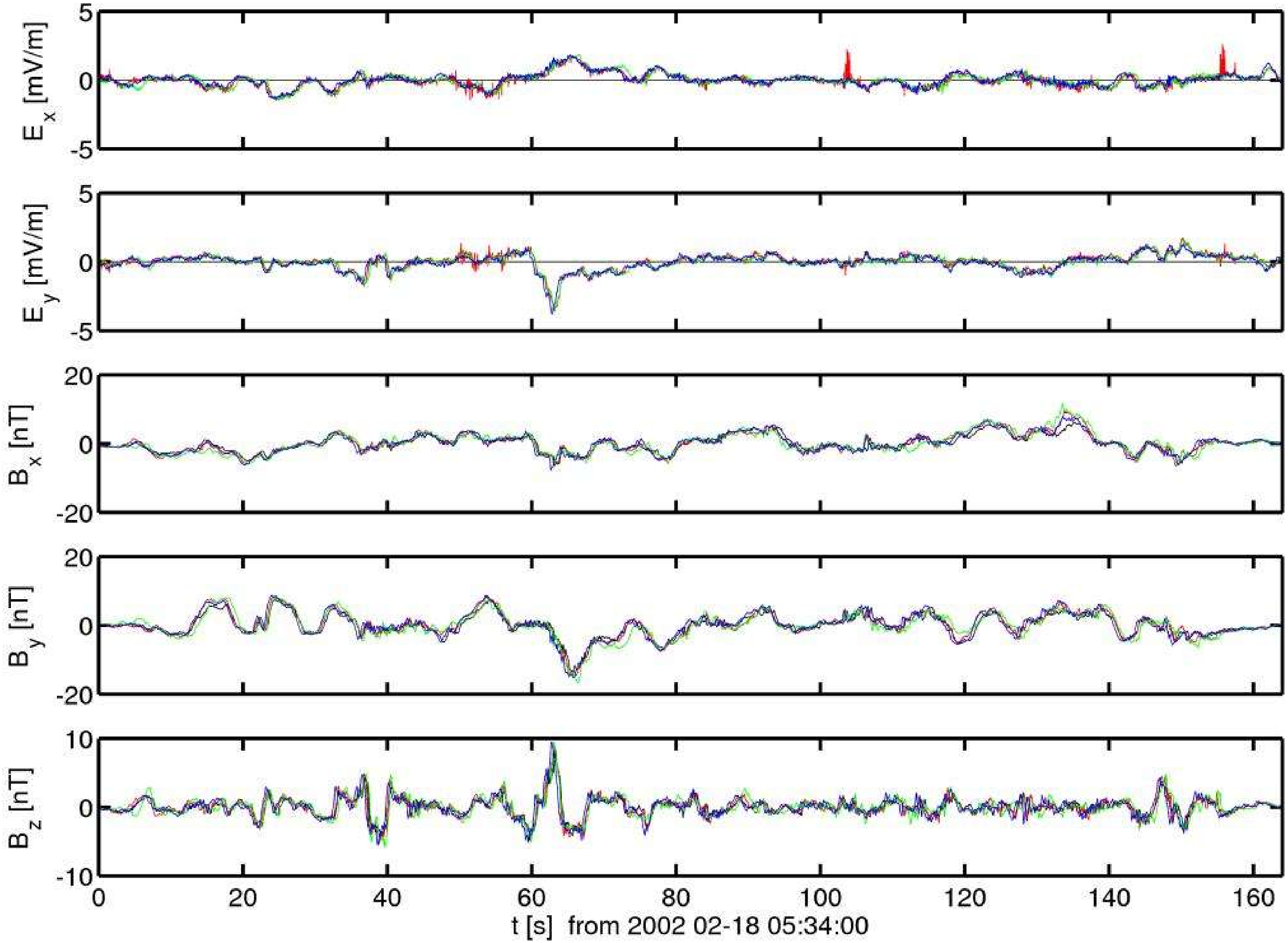


The spacecraft constellation is a nearly perfect tetrahedron with side 100 km



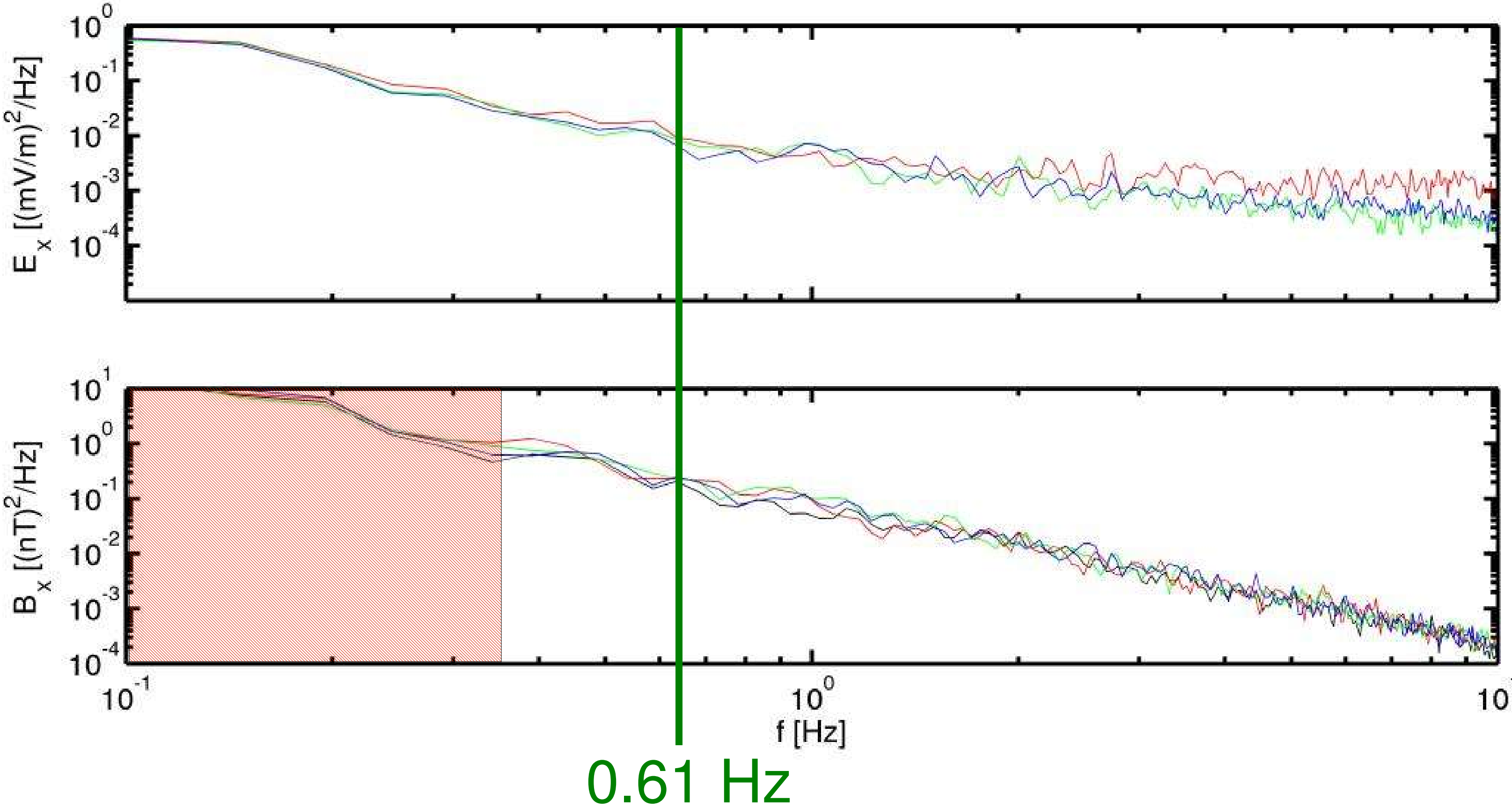
k-filtering example

The time series



k-filtering example

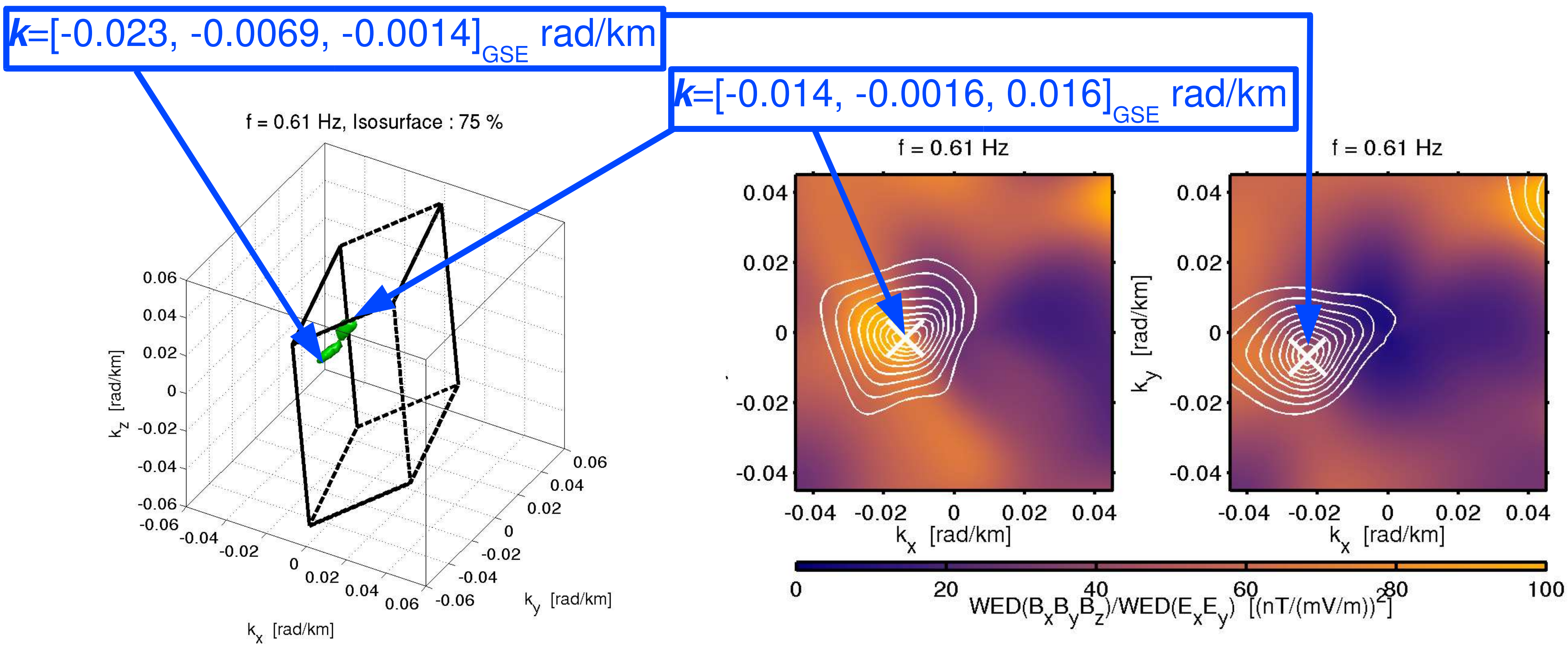
Spectra of the *x*-components of the electric and magnetic field on the four spacecraft.



k-filtering example

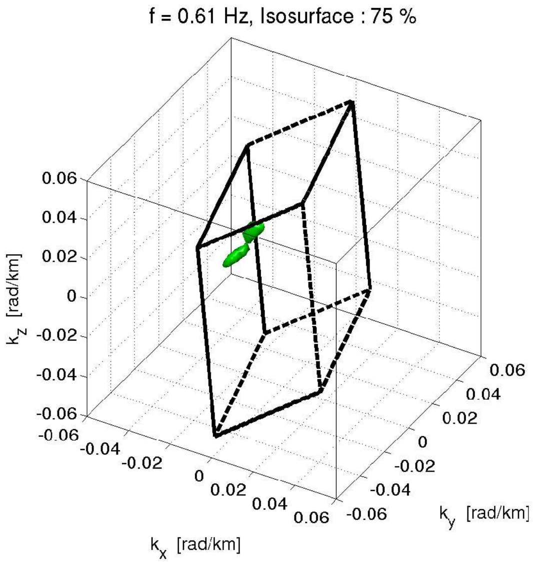
With the *k*-filtering technique it is possible to find spectral peaks at several wave vectors for each frequency.

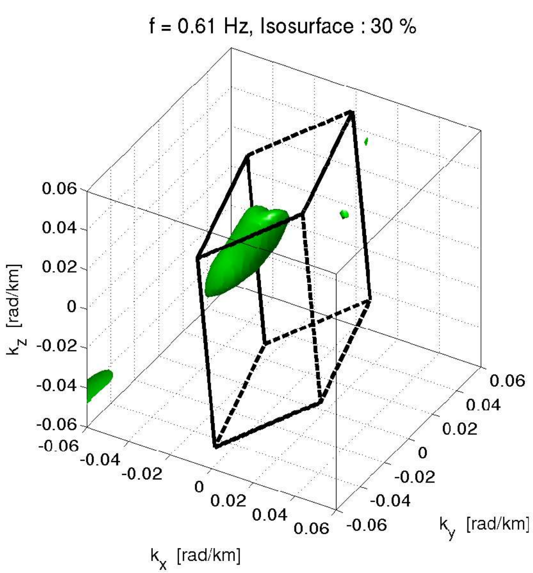
Below are two different visualisations of the wave-field energy distribution at 0.61 Hz.



k-filtering example

B and E





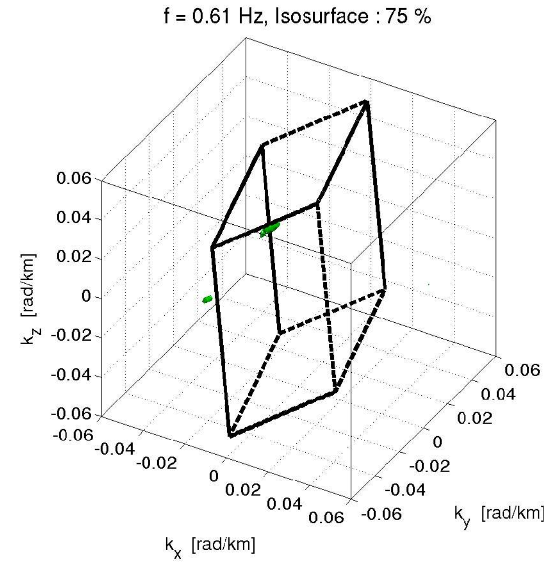
The inclusion of the electric field measurements gives us:

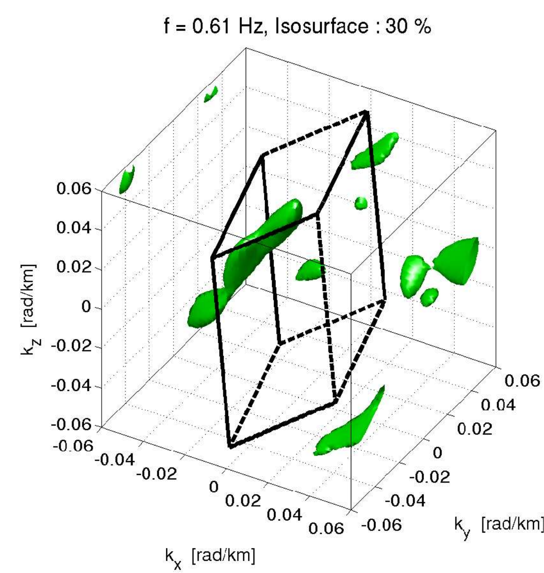
Possibility to analyse the wave polarisation

A small amount of extra noise

Large reduction of the effects from spatial aliasing

Only **B**





Conclusions

k-filtering has previously been using only magnetic field measurements

We have seen that *k*-filtering can be performed combining both electric and magnetic field measurements on the Cluster satellites

This gives us a possibility to enhance the resolution of the *k*-filtering technique

This also enables investigations of the relative importance of the electric and magnetic part of the wave-field energy distribution.