Requirements Specification

WEC/ISDAT Coordinate Transformations

System Functional Requirements
and
Architectural Design Considerations

Issue 1, Revision 7

WECdata Coordinate Systems Working Group

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Bars indicate changes since Issue 1, Revision 6, dated October 6, 2000

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1. Document Title: **WEC/ISDAT Coordinate Transformations**

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3. Issue	4. Revision	5. Date	6. Reason for Change
0	0	December 7, 1994	First draft
0	1	December 23,1994	Second draft, completely rewritten
0	2	December 29, 1994	Sect. 4.5 completed Sect. 4.7 added (and subsequently removed)
0	3	January 11, 1995	Paper entirely re-organised to: • include use of WEC/ISDAT structures • generalise for any coord. transformation • include first cut at implementation issues
0	4	January 22, 1995	Changes unknown
1	0	March 10, 1995	Chapter 1: substantially changed Sections 2.1 and 2.2: few, if any changes Section 2.3: new section. The old section 2.3 has moved to chapter 5 Section 2.4: new section Section 2.5: the former section 2.4 is unchanged, but only through lack of time Chapter 3: has been split into two sections: • spacecraft related coordinate systems (section3.1) and • geophysical coordinate systems (section 3.2) Section 3.1 is essentially unchanged Section 3.2 is new. It includes the headings of the former sections 3.2 through 3.6 (although these sections were not actually written in January) Chapter 4: unchanged. Section 4.3 needs updating to make it compatible with Chapter 5 Chapter 5: entirely rewritten. Figure 1 needs more work Table 3 is unchanged. It probably needs updating I have not printed the appendices, which are unchanged This ws the version on the WEC server until October 6, 2000

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3. Issue	4. Revision	5. Date	6. Reason for Change
1	1	October 22, 1995	
1	1	June 17, (?) 1996	Modifications to Table 1 Major changes to sect. 3.2, only text of sect. 3.2.3 remains Sect. 3.3.1 rewritten Elimination of the section "The overall transformation SC → GSE", which is adequately covered in the general Section 5.3
1	2	May 14, 1999	Conversion to LaTeX2
1	3	May 18, 1999	Correction of minor typos
1	4	October 12, 1999	*A pdf version of this revision was produced, dated April 10, 2000* Transformation of tensors appended to Sect. 2.1 Major clarification of Sect. 3.4, with corresponding updates to Tables 1 and 2 Correction of eqs. 48 and 49 Improvements & additions to Fig. 1
1	5	April 28, 2000	GSE changed to GEI twice in Sect. 2.1 Clarification of Sect. 3.4, including replacement $\hat{\mathbf{x}}^1$ by $\mathbf{x}'^1$ (which need not be normalised) New Section 3.4.4 Correction of minor typo in Sect. 4.1.4
1	6	October 6, 2000	Nomenclature of GSE corrected in Table 1. All references updated to Issue 3 of DDID. SATT file contents described once only, in Sect. 3.1.5. α_h, δ_h changed to α_p, δ_p in eqs. 15, 17, 43, 44 and 45 (coordinates of pole, not of Sun) Value of δ_p corrected in eq. 15. Simplification of Sect. 4.3 with development of Eqs. 44 and 45. Clarification of Sect. 4.5 with simplification of Eq. 49. Scalar products of Eq. 49 developed in Eq. 50. Eq. 51 gives an approximation for Eq. 49. Cluster 2 data used partly for Appendix 3 *(but further work is needed).*
1	7	May 3, 2001	Sense of rotation corrected in eq. 17, and consequences of this error corrected in • in all three of eqs. 50 • in two of eqs. 62 Inclusion of Inverted Despin system (IDS): • IDS inserted in Tables 1 and 2 • new last paragraph of Sect. 3.1.6 • new Sect. 3.1.7 • new Sect. 4.6 including 3 subsections

1 Introduction

WEC/ISDAT has a number of "known" reference coordinate systems. These are coordinate systems which are recognised by ISDAT. Any non-scalar information, such as field direction, sensor axis direction, or sensor location, must be defined with respect to one of these "known" coordinate systems.

The known system with respect to which nearly all spacecraft experiment sensors are stationary is the spacecraft mechanical build coordinate system which, as its name implies, is the coordinate system used to build the spacecraft. The only experiments which do not telemeter data in a coordinate system fixed with respect to the mechanical build system are experiments mounted on platforms (inapplicable in the case of Cluster), and experiments which perform a coordinate transformation aboard the spacecraft (*e.g.*, the Cluster STAFF Spectrum Analyser).

The science data analysis generally requires the data to be represented in some coordinate system which is not related to the spacecraft. Depending upon the analysis being performed, it may be related to the Earth, the terrestrial magnetic field, the position of the Sun, or a combination of these factors.

The purpose of this document is to define methods of transforming data from one coordinate system to another and to show how they can be implemented using the WEC/ISDAT architecture. These transformations apply only to calibrated data, in the sense that all elements of any vector or tensor quantity must be expressed on the same linear scale of units.

A prerequisite is to define the different coordinate systems used and, where appropriate, compare them with systems used in other Cluster-related documents. The systems discussed in this document are summarised (and their acronyms shown) in Table 1 ; they are be defined in Section 3. The transformations between these different transformations can be divided into two classes:

1. Transformations specific to Cluster. These convert experiment coordinates to an inertial coordinate system; the inertial system chosen here is the geocentric equatorial inertial (GEI) system. The information presented here is based upon information extracted from references 1 through 4. Note that the transformation from the sensor coordinate system, explained in section 4.1.1, has the peculiarity of being, in general, non-orthogonal.

   Some of these transformations are time dependent. Some transformations (SR $\rightarrow$ DS) vary rapidly with time and the rotation matrix must be constructed from the satellite housekeeping; others (AS $\rightarrow$ SR and DS $\rightarrow$ GEI) vary more slowly and use data from the Spacecraft Attitude (SATT) file. The rest are constant in time and may either differ (SC $\rightarrow$ WEC) or be the same (MB $\rightarrow$ AS) for the four spacecraft, but still specific to the Cluster mission.

2. More general transformations, which are standard: that is, they apply to data from any spacecraft. These transformation are explained in refs. 5 and 11, and the definitions are merely copied to this document, so that the implementation for WEC/ISDAT can be described. Furthermore, the software developed in conjunction with ref. 5 will be used wherever possible, as explained in Section 5.3.

   These transformations too may be constant in time, or have an annual variation, a diurnal variation, or both, as shown in Table 2.

The data structure used for WEC/ISDAT version 2.0 (ref. 6) has been designed to simplify coordinate transformation, in the sense that all the information required is present in the data structure (see section 2.3), and the data itself is organised in a standard way to facilitate the use of generic software. Nevertheless, the treatment of coordinate transforms is still not easy: there are so many possible transformations ! Precisely for this reason, a major effort is made to define a generic coordinate transformation operator.

WEC This document		ESTEC refs. 1 & 2	ESOC ref. 3	JSOC ref. 4	FGM ref. 8	*
Rotating systems						
Sensor Coordinates	SC	–	–	–	non-orthogonal†	$\mathbf{s}^i$
WEC	WEC	–	–	–	–	$\mathbf{w}^i$
Mechanical Build	MB	mech. build	body-build	–	mech. build	$\mathbf{b}^i$
Attitude System	AS	–	attitude	–	–	$\mathbf{a}^i$
Spin Reference	SR	–	spin reference	SR1	–	$\mathbf{r}^i$
Near-Inertial system						
Despun Satellite	DS	–	–	SR2	NSS‡	$\mathbf{d}^i$
Inverted Despun	IDS	–	–	-	-	$\mathbf{i}^i$
Inertial system						
Geocentric Equatorial Inertial	GEI	–	–	GEI	GEI	$\mathbf{e}^i$
Scientific coordinate systems						
Ecliptic	ECL	–	–	–	–	$\mathbf{h}^i$
Geocentric Solar Ecliptic	GSE	–	–	–	–	$\mathbf{g}^i$
Data defined coordinate systems						
Magnetic field		–	–	–	–	$\mathbf{x}^i$
Minimum variance						
Boundary normal	LMN					
etc.						

* Symbols used to represent the base vectors in chapter 4.
† This system is aligned with the FGM (not the WEC !) sensors.
‡ This system has its axes permuted so as to spin about the O1 axis.

Table 1: Satellite-related coordinate systems used here and elsewhere.

2 Coordinate Transformation

2.1 Mathematical Foundations

The symbols used in this technical note to represent base vectors of each coordinate system are shown in the last column (headed by an asterisk) of Table 1. Thus the base vectors of, for example, the spin reference system are $\mathbf{r}^i$, for $i = 1$, 2 or 3. In this coordinate system, the magnetic field $\mathbf{B}$ is represented by the three components B_i^r,

$$B_1^r = \mathbf{r}^1 \cdot \mathbf{B} \quad , \quad B_2^r = \mathbf{r}^2 \cdot \mathbf{B} \quad \text{and} \quad B_3^r = \mathbf{r}^3 \cdot \mathbf{B} \ . \tag{1}$$

Similarly, in the GEI coordinate system the components B_i^e of the $\mathbf{B}$-field are

$$B_1^e = \mathbf{e}^1 \cdot \mathbf{B} \quad , \quad B_2^e = \mathbf{e}^2 \cdot \mathbf{B} \quad \text{and} \quad B_3^e = \mathbf{e}^3 \cdot \mathbf{B} \ . \tag{2}$$

The base vectors $\mathbf{r}^i$ of the spin reference system may be represented in their own coordinate system,

$$\mathbf{r}^i \cdot \mathbf{r}^j = \delta_{ij} \tag{3}$$

or, in the GEI system,

$$\mathbf{e}^i \cdot \mathbf{r}^j = R(er)_{ij} \ . \tag{4}$$

Here $R(er)_{ij}$ is a matrix with each element i, j being the projection of the base vector $\mathbf{r}^j$ onto the base vector $\mathbf{e}^i$. Clearly,

$$R(er)_{ij} = R(re)_{ji} \ . \tag{5}$$

It may be shown (Appendix 1, eq. 57) that

$$\sum_{i=1}^{3} \mathbf{r}^i \mathbf{r}^i = \mathcal{I} \tag{6}$$

where $\mathcal{I}$ is the unit operator. Although the signification of this equation is perhaps conceptually difficult, its representation in Cartesian coordinates (eq. 10) is quite familiar. Equations 3 through 6 may be slightly modified to generalise them to non-orthogonal systems (see Appendix 1).

Equations 3 through 6 may be used to:

1. Express the components B_i^e in terms of the components B_i^r and the matrix $R(dr)_{ij}$

$$B_i^e = \mathbf{e}^i \cdot \mathbf{B} = \mathbf{e}^i \cdot \sum_{j=1}^{3} \mathbf{r}^j \mathbf{r}^j \cdot \mathbf{B} = \sum_{j=1}^{3} (\mathbf{e}^i \cdot \mathbf{r}^j) \ (\mathbf{r}^j \cdot \mathbf{B}) = R(er)_{ij} \ B_j^r \ . \tag{7}$$

   Thus we see that the matrix $R(er)_{ij}$ may be used to convert the representation of a vector $\mathbf{B}$ from the $\mathbf{r}^i$ system of coordinates to the $\mathbf{e}^i$ system of coordinates. This is loosely termed "rotation of the vector $\mathbf{B}$" although, of course the vector itself is unchanged, only its representation changes.

2. Show how to combine any two consecutive rotations to form one single rotation.

$$\mathbf{e}^i \cdot \mathbf{r}^j = \mathbf{e}^i \cdot \sum_{k=1}^{3} \mathbf{d}^k \mathbf{d}^k \cdot \mathbf{r}^j = \sum_{k=1}^{3} (\mathbf{e}^i \cdot \mathbf{d}^k) \ (\mathbf{d}^k \cdot \mathbf{r}^j) \tag{8}$$

   so that

$$R(er)_{ij} = R(ed)_{ik} R(dr)_{kj} \ . \tag{9}$$

3. Derive the relation between a rotation and its inverse.

$$\sum_{k=1}^{3} (\mathbf{e}^i \cdot \mathbf{r}^k)\ (\mathbf{r}^k \cdot \mathbf{e}^j) = \mathbf{e}^i \cdot \sum_{k=1}^{3} \mathbf{r}^k \mathbf{r}^k \cdot \mathbf{e}^j = \mathbf{e}^i \cdot \mathbf{e}^j = \delta_{ij}$$

so that (and using eq. 5)

$$R(er)_{ik} R(re)_{kj} = \delta_{ij} = R(er)_{ik} R(er)_{jk}\ . \tag{10}$$

A second rank tensor quantity, such as a correlation $\mathbf{C}$, may be represented by the nine quantities (*c.f.* eq. 1)

$$B^r_{ij} = \mathbf{r}^i \cdot \mathbf{C} \cdot \mathbf{r}^j\ . \tag{11}$$

The transformation of a tensor quantity is analogous to that of a vector (*c.f.*, eq. 7)

$$\begin{aligned} C^e_{ij} &= \mathbf{e}^i \cdot \mathbf{C} \cdot \mathbf{e}^j \quad = \quad \mathbf{e}^i \cdot \sum_{k=1}^{3} \mathbf{r}^k \mathbf{r}^k \cdot \mathbf{C} \cdot \sum_{\ell=1}^{3} \mathbf{r}^\ell \mathbf{r}^\ell \cdot \mathbf{e}^j \\ &= \sum_{k=1}^{3} \sum_{\ell=1}^{3} (\mathbf{e}^i \cdot \mathbf{r}^k)(\mathbf{r}^k \cdot \mathbf{C} \cdot \mathbf{r}^\ell)(\mathbf{r}^\ell \cdot \mathbf{e}^j) \\ &= R(er)_{ik}\, R(re)_{\ell j}\, C^r_{k\ell} \quad = \quad R(er)_{ik}\, R(er)_{j\ell}\, C^r_{k\ell} \quad , \end{aligned} \tag{12}$$

where eq. 5 has been used.

Similarly, a third rank tensor, for example the heat flux $\mathbf{H}$, transforms thus :

$$H^e_{ijk} = R(er)_{i\ell}\, R(er)_{jm}\, R(er)_{kn}\, H^r_{\ell mn}\ . \tag{13}$$

2.2 Integration into the WEC/ISDAT Structure

The expressions of the preceding section allow coordinate transforms to be integrated naturally into the WEC/ISDAT data structure described in ref. 6. Comparison of eqs. 1 and 4 shows that the rotation matrix $R(er)_{ij}$ is simply the representation of the base vectors $\mathbf{r}^j$ of the "old" coordinate system in the "new" coordinate system with based vectors $\mathbf{e}^i$. Thus the rotation matrix $R(er)_{ij}$ is a perfectly valid logical instrument, which may be described following the proforma menu of chapter 3 of ref. 6, as follows.

2.2.1 General

1.	Nature of the parameter being measured:	"old" coordinate axis
2.	Units in which it is measured:	none
3.	Type:	real
4.	Numerical "fill" value:	??
5.	Flight instrument sampling frequency:	n/a
6.	Cut-off frequency of the flight instrument pre-detector low-pass filter:	n/a
7.	Oversampling factor:	TBD ? (very large)
8.	Version number of the programme and the calibration software used to produce the data:	??
9.	Sensor location with respect to the origin of coordinates:	n/a
10.	The known WEC coordinate system in which this sensor location is expressed:	n/a
11.	A flag to indicate if the data is in telemetry units:	n/a
12.	Name and e-mail address of contact person:	??
13.	Name of the PI:	n/a
14.	Other information, still TBD, required by WEC or by CDF:	??

NOTE. During a meeting between AA, GH, JZ, CdeV and CH in ESTEC on 1995 February 23, uncertainty was expressed concerning the necessity of describing a coordinate transformation in terms of a logical instrument with the standard WEC data structure. Work, stalled since 1996 June 04, is now ongoing

2.2.2 Timing

The time of the data samples is specified by

- The time of the first data sample,
- the time interval covered (i.e., the difference between the times of the last and the first data samples), and
- the number of samples.

For uniformly sampled data, this is all that is required; such data is called "segmented" data.

Transformations between rotating coordinate systems are not constant, and so the data object which represents the transformation must consist of a time series of data bricks (see ref. 6) with sufficient (time) resolution to permit the determination of the instantaneous matrix by joining. Even for a non-varying coordinate transformation, to permit joining it is preferable that the data object which represents the transformation contain at least two data bricks, one at the start and one at the end of the overall data interval. Time resolution is discussed further in section 2.4.

2.2.3 Dimension

1.	Dimension:	1
2.	Physical nature of the domain:	change of coordinate base vector
3.	Physical units used in the domain:	n/a
4.	"Number" of discrete values in the domain:	3
5.	Map:	1, 2, 3 (*i.e.*, the labels of the coordinate axes)
6.	"Offset" table:	0, 0, 0
7.	"Window" table:	n/a
8.	Lower limit of the plot scale:	−1
9.	Upper limit of the plot scale:	+1
10.	Most suitable plot scale:	linear

2.2.4 Rank

1.	Data brick type:	complete
2.	Rank:	1
3.	Reference coordinate system:	the "new" coordinate system
4.	Orientation of sensor axes:	perfect alignment (use default option of the unit matrix)

2.3 Relation to the Data Structure

It is clear that the logical instrument described above is closely related to the information required for the WEC/ISDAT data structure, under item 4 of section 3.4 (ref. 6). This is not surprising. The data structure is designed to be fully self-describing, and therefore if data from any vector or tensor logical instrument is not in one of the known WEC coordinate systems, the metadata must provide the information required to transformation to one of the known WEC systems. The only alternative would be for WEC/ISDAT to "know" *a priori* every coordinate system likely to be used, including systems related to sensors and to external data sets. This is clearly impossible, and so every data object delivered in an unknown coordinate system would need WEC/ISDAT to be modified to make the data usable.

The WEC/ISDAT data structure requires every data object to include within its structure the matrix required to transform the data to a known WEC coordinate system. The form in which this matrix is presented is formally identical to the representation of coordinate transformation as a logical instrument as described in the preceding section.

This fact may be taken into account and used by the coordinate transformation operator (*still TBD*).

2.4 Time-varying Coordinate Transformations

Coordinate systems can be divided into different categories according to their motion with respect to a truly inertial coordinate system:
1) stationary (for all practical purposes);
2) slowly rotating (one rotation per year);
3) rapidly rotating (one rotation per day);
4) despin (for Cluster, one rotation every 4s);
5) irregular motion (*e.g.*, the satellite spin axis direction);
6) systems defined by the data itself (*e.g.*, boundary normal coordinates).

A changing coordinate system can be handled by creating a time series for the logical instrument representing the rotation (just like any other logical instrument), then joining this time series to the physical data to be transformed in order to obtain an appropriate instantaneous value for the transformation matrix.

To "join" the rotation matrix to the physical data, the "logical instrument" which describes the rotation must consist of a series of data objects which are (probably) uniformly spaced in time with adequate resolution for joining. To determine the required time resolution, we note that to obtain an angular precision of 1° when using a sloppy join, the following time resolution would be required respectively for each of the above four characteristic rotational times scales:
1) years
2) 1 day
3) 4 mn
4) 11 ms.

The use of linear interpolation for the joining algorithm reduces the angular imprecision to a negligible value, but introduces another problem: linear interpolation of a evolving unitary transformation does not yield a unitary transformation. To examine this effect, let us consider the vector obtained by linear interpolation (linear joining) between two unit vectors inclined at an angle θ. The resulting vector is simply the average of the two tabulated vector, and it has length $\cos(\theta/2) \simeq 1 - \frac{1}{8}\theta^2 + \ldots$. The fractional length error is less than $\frac{1}{8}\theta^2$, and for an angular separation of 1°, this is less than 4×10^{-5}. The introduction of such a small error onto the magnitude of experimental vector data is considered acceptable.

NOTE. At a meeting with Joe Zender in ESTEC on 1995 February 23 it was suggested that, rather than use interpolation, it may be better to calculate a rotation matrix to coincide with every data point in the data file. Note, however, that for a simple vector quantity, the transformation matrix file (of nine element matrices) will be three times the size of the actual data file (of three element vectors). Thus the system will be limited by the transformation matrix, not by the data, and interpolation would increase the maximum acceptable length of vector data files by a factor of approximately four !

2.5 Two-dimensional Vectors

Two-dimensional vectors can be rotated only in the plane of their two components. The only coordinate transformation of any significance is the despin operation of the preceding section. If it is thought desirable to use the permutation of axes described in section 4.1.3, special modification will be required. The realisation of these operations is described in Chapter 5.

3 Definitions of Coordinate Systems

All coordinate systems are right-handed and all except the sensor coordinate system are orthogonal.

Most of the inertial coordinate systems, and the associated transformations, are discussed at length in references 11 and 5.

3.1 Spacecraft Related Coordinate Systems

Everyone who has studied coordinate systems associated with spinning spacecraft must necessarily use the same or closely related coordinate systems, but not necessarily with the same name. Table 1 shows the nomenclature used elsewhere for the basic coordinate systems discussed in this document.

A satellite, like any other object, can spin in a stable manner only about its axis of maximum moment of inertia. When constructing a satellite designed to spin, great care is taken to ensure that the direction of maximum moment of inertia of the spacecraft is close to one of the mechanical build coordinate axes. Fine trimming is performed by attaching small masses to appropriate parts of the spacecraft structure until the relative alignment of the spin-axis and mechanical build axes is within the permitted design tolerance, as confirmed by balance (to determine the centre of gravity) and dynamic balance (to determine the axes of inertia) tests.

WEC/ISDAT can transform data from any coordinate system to any other coordinate system, by specification of the transformation matrix. However, for a spinning spacecraft, any matrix to transform from a spinning to an inertial system is itself changing as fast as the spacecraft is spinning. For this reason, rather than frequently calculate the complete transformation matrix, it is preferable to proceed by successive simple rotations as explained in section 5.3. Such a policy of successive simple rotations is also recommended by Hapgood (ref. 4).

The coordinate systems presented in this section are required by WEC/ISDAT.

3.1.1 Sensor Coordinates (SC)

Mechanically, the search coil axes are closely aligned (better than 0.5°) with the long wire antennas; nevertheless, their electrical axes are slightly offset from their mechanical axes, and this effect must be described in the attributes of the corresponding logical instruments. As described in ref. 6, as part of the "data structure" the direction cosines of each axis the logical instrument must be specified with respect to one of the "known" WEC coordinate systems. In the case of the WEC sensors, the appropriate known system is the spacecraft mechanical build system.

The orientation of the STAFF Search Coil axes with respect to the WEC coordinate system is presented in Appendix 3.

3.1.2 WEC Coordinates (WEC)

The WEC long wire antennas are in the spin plane at ±45° with respect to the mechanical build axes. This gives rise to the "WEC" coordinate system, the axes of which are defined in terms of the spacecraft mechanical build axes in the EID–B (ref. 2), Section 2, page 100, Fig. 2.6/1, WEC reference axis:

$$\mathbf{w}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \mathbf{w}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \mathbf{w}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad . \tag{14}$$

3.1.3 Mechanical Build Coordinates (MB)

This is the coordinate system used to construct the spacecraft, as defined in ref. 1 on page 5 of Section 2. In ref. 3 it is called the "body-build" coordinate system. Normally a spinning spacecraft is constructed so that the principal axes of inertia (the eigenvectors of the inertia tensor) are close to the axes of the mechanical build coordinate system.

1. For Cluster the mechanical build O1 axis has been defined as the nominal spin axis (ref. 1, *bid*), with the main motor in the –O1 direction. The radial rigid booms are along the O2 axis, with radial boom 2 (which supports the STAFF search coil) in the +O2 direction.
2. For many spacecraft (*e.g.*, ISEE, Ulysses), O3 is defined to be the nominal spin axis with the Ox and Oy axes in the nominal spin plane.

3.1.4 The Attitude System (AS)

This is a system of axes aligned with the mechanical build system, but with the axes permuted so that the O3-axis is parallel to the nominal spin axis (ref. 3, page 136).

The purpose of this coordinate system is to facilitate the maintenance of generic software, that is, software which is maintained and applied to data from several different projects. This is presumably why ESOC introduces the system, and it is similarly useful for ISDAT.

3.1.5 The Spin Reference system (SR)

Once launched, the spacecraft spins stably about its axis of maximum inertia. The spin reference coordinate system is defined (ref. 3, page 136) to be an orthogonal Cartesian system with the O3 axis passing through the satellite centre of mass in the direction parallel to the spin axis as determined in-flight, and the O1 axis in the O13 plane of the attitude system (or, equivalently, the O12 plane of the mechanical build system).

Although the spin reference system is nearly aligned with the mechanical build and attitude systems, it is nevertheless conceptually different: the mechanical build system is defined during and for the construction of the spacecraft, while the spin reference system is defined by the operational spin axis, as determined by in-flight measurements. If the dynamical balance is good and all booms deploy correctly, the attitude and spin reference coordinates systems may be considered to be equivalent (see section 4.1.4).

The spin axis is determined in flight using data from the solar sensors, and the following information is distributed on CD-ROM in the Satellite ATTitude (SATT) file which is described in Appendix E.5 on page 88 of the Data Delivery Interface Document (DDID, ref. 3).

1. The shift (in millimetres) of the spacecraft centre of mass with respect to the origin of satellite mechanical build coordinates is to be found in the vector COMSHF.

2. The Euler angles Ψ_1 and Ψ_2 (in degrees) for the transformation from attitude to spin reference coordinates are to be found in the variables TPSI_1 and TPSI_2.

3. The spin axis direction $\mathbf{x}$ expressed in celestial (Geocentric Equatorial Inertial) coordinates in terms of its right ascension α_x and declination δ_x (for epoch J2000.0) are supplied by the variables SPRASC and SPDECL.

3.1.6 The Despun Satellite system (DS)

The despun reference system is an orthogonal Cartesian coordinate system with the O3 axis along the spacecraft spin axis (and thus coinciding with the O3 axis of the spin reference system) and the O1 axis in the plane containing this spin axis and the direction of the Sun. The spin axis direction can be found in the SATT file (Point 3 of preceding section). Relative to a “truly” inertial system, the despun satellite system rotates about its O3 axis with a period of one year. In addition, the direction of the O3 axis changes discontinuously each time the spacecraft performs a manoeuvre.

Many spin-stabilised spacecraft, such as ISEE, have their spin axis closely closely aligned with the celestial pole ; then the DS system is often close enough to the GSE system to be useful for understanding the physics. Transformation to this system simpler than to the geophysical systems discussed in Section 3.3 and is, in any case, a necessary intermediate step. Furthermore, some “incomplete” vector quantities which are measured only in the spin plane (such as the electric field) can be transformed to DS coordinates, but cannot be transformed to GSE or other geophysical coordinates systems.

3.1.7 The Inverted Despun Satellite system (IDS)

Some spin-stabilised spacecraft, including Cluster, have their spin axis pointing near to the south ecliptic pole. Then the DS system is close to being rotated through 180° with respect to the GSE system. For these spacecraft it is useful to introduce the Inverted Despun System (IDS), which is simply the DS rotated through 180° around the O1 axis which is common to both systems. For these spacecraft, the IDS system is close to the the GSE system.

3.2 Inertial Coordinate Systems

The coordinate systems presented in this section are used for dynamical analysis because they are close to a truly inertial system. The Galactic coordinate system is the “most inertial”, but its use is somewhat inconvenient, for example, for visualising orbits. Therefore a system defined by reference to the Earth’s rotation axis is used, the Geocentric Equatorial Inertial coordinate system being the most frequently used system. But there is a price to pay: the precession of the Earth’s rotation axis, which has several components of which the most significant has a period of 26,000 years, means that the coordinate system is slowly changing. To have systems which are truly inertial, the GEI system is “frozen” at different times. The epoch 1950 system was used until the early 1980’s, now the epoch 2000 coordinate system is the most widely used, although sometimes “epoch of date” (i.e., frozen today) coordinate systems are encountered. For further details, see Hapgood (ref. 14).

3.2.1 Geocentric Equatorial Inertial (GEI)

The GEI coordinate system is the principal inertial coordinate system, that is, the one in terms of which all the other systems are defined.

It has its origin at the centre of the Earth. The O3 axis is defined by the Earth's rotation axis, and the O1 axis by the First Point of Aries (♈), which is the intersection of the Earth's equatorial plane with the plane of ecliptic (the point at which the Sun passes from the southern into the northern celestial hemisphere). The O2 axis completes the right-handed triad.

3.2.2 Ecliptic (ECL)

This system is also geocentric, with its O3 axis directed towards the north ecliptic pole, which has GEI coordinates

$$\begin{array}{rrcll} \text{right ascension} & \alpha_p & = & 18^h\,00^m & \text{(by definition)} \\ \text{declination} & \delta_p & = & 66^\circ\,33'\,15'' & \text{(epoch 1950)} \end{array} \tag{15}$$

[It is more conventional in the astronomical community to talk about the obliquity of the ecliptic, which is $23^\circ\,26'\,45''$.] The origin of longitude coincides with the origin of longitude of the GEI system. Therefore the transformation from GEI to ECL coordinates is

$$x_i^h = R(he)_{ij}\, x_j^e \tag{16}$$

where

$$R(he)_{ij} = \mathbf{h}^i.\mathbf{e}^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\delta_p & \cos\delta_p \\ 0 & -\cos\delta_p & \sin\delta_p \end{pmatrix} \tag{17}$$

3.2.3 Galactic (GAL)

The new (1959) system of galactic coordinates is defined with the O3 axis in the direction of the north galactic pole, which has celestial coordinates

$$\begin{array}{rrcll} \text{right ascension} & \alpha_g & = & 12^h 49^m & \\ \text{declination} & \delta_g & = & 27^\circ\,40' & \text{(epoch 1950)} \end{array} \tag{18}$$

and the meridian plane containing the O1 axis passes through the centre of the Galaxy; then the equatorial north pole (*i.e.*, of GEI coordinates) has Galactic longitude $l^{II} = -123^\circ$.

3.3 Geophysical Coordinate Systems

The coordinate systems presented in this section are defined for their convenience in geophysics.

Each coordinate system requires three angles to define it completely. Two angles are required to define the direction of one of the coordinate axes. The other two axes are mutually orthogonal, and their orientation about the first axis is defined by the third angle; this is conveniently expressed in terms of a vector which defines the meridian plane containing one of these axes.

Ten commonly used geophysical or astronomical coordinate systems are defined in this way in Table 2. Most of the information in this table has been copied from reference 5. The first column gives the name of the system and its common acronym. The second column indicates the coordinate axis defined by its direction, and defines that direction. The last column indicates the coordinate axis defined by a meridian plane, and specifies the vector used to define that plane. The various coordinate systems are listed in Table 2 according to their rate of variation with respect to a truly inertia system. The first two coordinate systems are the same as the last two of Table 1; they are of special importance for Cluster.

Further information concerning the use or definition of these different coordinates systems is given in the following sections.

3.3.1 Annual motion

Geocentric Solar Ecliptic (GSE)

This system is defined to have the same O3 axis as the ecliptic system (the direction of the north ecliptic pole), but with the O1 axis in the direction of the Sun. The longitude ϕ_h of the Sun in the ecliptic coordinate system (*i.e.*, the heliocentric longitude of the Sun) may be computed from the motion of the Sun along the ecliptic (*e.g.*, from Keppler's equations, as in ref. 13, pages 121-126).

The transformation from ECL to GSE coordinates is therefore

$$x_i^g = R(gh)_{ij}\, x_j^h \tag{19}$$

where

$$R(gh)_{ij} = \mathbf{g}^i.\mathbf{h}^j = \begin{pmatrix} \cos\phi_h & \sin\phi_h & 0 \\ -\sin\phi_h & \cos\phi_h & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

Geocentric Solar Equatorial (GSEQ).

This system is defined to have its O1 axis in the direction of the Sun, and its O3 axis perpendicular to the Sun's equatorial plane.

3.3.2 Annual & diurnal motions

The following coordinate systems have both an annual and a daily motion with respect to a truly inertial system.

Geocentric Solar Magnetic (GSM).

Solar Magnetic (SM).

3.3.3 Diurnal motion

The following coordinate systems have daily motion with respect to a truly inertial system.

Geographic (GEO).

Geomagnetic (MAG).

3.3.4 Local terrestrial systems

The following coordinate systems depend upon the coordinates of the point of observation on the Earth's surface.

Dipole Meridian (DM).

Vertical Dusk Horizontal (VDH).

3.4 Data-dependent systems

These coordinate systems are related to the science data, and therefore vary on a time-scale decided by the scientist as he performs his analysis. The scientist must first define the axes of the coordinate system in which he wants to express the data. A convenient way of doing this is to define :

- The direction $\mathbf{x}^3$ of the O3-axis ;
- The phase angle of the O1-axis. The most general way of specifying this is to define a second direction $\mathbf{x}'^1$ which, together with 03, defines the plane containing 01. The triad is then defined by

$$\begin{array}{rcl} \mathbf{x}^3 & & \\ \mathbf{x}^2 & = & (\mathbf{x}^3 \times \mathbf{x}'^1)/\sqrt{1-(\mathbf{x}^3\cdot\mathbf{x}'^1)^2} \\ \mathbf{x}^1 & = & (\mathbf{x}'^1 - (\mathbf{x}^3\cdot\mathbf{x}'^1)\mathbf{x}^3)/\sqrt{1-(\mathbf{x}^3\cdot\mathbf{x}'^1)^2} \end{array} \tag{21}$$

The transformation of the data to the new data-dependent coordinate system is trivial, via the matrix R_{ij} of eq. 7. For example, if both the data and the reference triad are expressed in the despun satellite system, then the three components $B_i^d = \mathbf{d}^i \cdot \mathbf{B}$ and and the nine elements $R(dx)_{ij} = \mathbf{d}^i \cdot \mathbf{x}^j$ are all known, and the transformations of vector and tensor quantities are simply (*c.f.* eqs. 7, 12 and 13)

$$B_i^x = R(dx)_{ij}\, B_j^d \tag{22}$$

$$C_{ij}^x = R(dx)_{ik}\, R(dx)_{j\ell}\, C_{k\ell}^d \tag{23}$$

$$H_{ijk}^x = R(dx)_{i\ell}\, R(dx)_{jm}\, R(dx)_{kn}\, H_{\ell mn}^d \,. \tag{24}$$

Examples of frequently used data-dependent coordinate systems are given in the following three subsections. The determination of the data-dependent coordinate system (*i.e.*, the determination of the triad $\mathbf{x}^i$) is a delicate task which requires careful consideration of both the data (which physical parameters to use), its processing (is any filtering required) and the time interval to be used for the determination. These points are discussed, for example, in the ISSI book. In any case, the end result is a definition of the coordinate system in terms of a reference triad of three axes $\mathbf{x}^i$ (Table 1) specified with respect to one of the "known" systems of reference. If the reference triad is obtained in an "unknown", coordinate system, it must be converted to a known system using eq. 8.

3.4.1 Magnetic coordinates

The mean magnetic field imposes an axis of symmetry on both wave and particle phenomena and, in a homogeneous medium, is helpful to represent vector and tensor quantities in a coordinate system related to the mean ambient magnetic field ; this is taken to be the O3 axis. The orientation around this axis is generally less important, but must nevertheless be specified by a second direction, which may be anything, chosen to suit the needs of the scientific investigation ; for examples : the direction of the Sun, the direction of plasma flow, the local magnetic meridian or geographic meridian or, at low altitudes, the local nadir.

3.4.2 Minimum variance coordinates

Minimum variance analysis is used to estimate, for example, the direction of propagation of a circularly polarised wave, or the normal to a rotational discontinuity. Minimum variance coordinates have the O3-axis in the direction of minimum variance and the O1-axis in the direction of maximum variance.

3.4.3 Boundary normal coordinates

In a strongly inhomogeneous medium it is more useful to represent vector and tensor quantities in a coordinate system related to the inhomogeneities, and boundary normal coordinates are used to study naturally occurring boundaries, such as the bow shock or magnetopause. The method used to determine the boundary is a matter of scientific choice ; for example, methods used for determining the normal to the Earth's bow shock include use of a statistical model of the shock, co-planarity, and optimisation of the Rankine-Hugoniot relations across the shock. Often the results of all three methods are compared before the "best" shock normal is defined. The O3-axis is defined to be the direction of boundary normal, and the second direction (used to define 01) is open to discussion; it may be the direction of maximum variance, or of plasma flow of the magnetic field on one side of the boundary, *etc.*

3.4.4 Determination of data dependent coordinate systems

How can the data-dependent coordinate system be determined: what data, and what algorithms, should be used to determine O3 and O1′ ? This is a scientific question, which is also somewhat subjective. The answer will vary from case to case, and the user must be able to specify his requirements. To allow this in the spirit of ISDAT, before performing data-dependent transformations the coordinate transformation module must generally open a pop-up window in the Graphical User Interface though which the user may enter his requirements. This is a whole new dimension for the coordinate transformation module, and explains why the issue has not been investigated earlier.

Whatever data is used to determine the coordinate system (magnetic field, particle moments, or other parameters), it is most likely that some sort of averaging or other processing of the data will be required to determine O3 and O1′; instantaneous data samples will not be used, unless averaging has already been performed. For example, consider conversion to a coordinate system defined by the mean magnetic field in a statistically homogeneous medium. The expression "mean magnetic field" is loosely used to designate the instantaneous value of the spectral component of the magnetic field varying on a time-scale long compared to the period of the parameter whose fluctuations are to be studied, but short compared to some other time scale which must itself be shorter than the characteristic interval between successive magnetic field discontinuities (which invalidate statistical homogeneity). Neither of these limits is clearly defined.

There are three essential questions which must be answered via the pop-up window:

1. what parameters are to be used,
2. during what interval of time, and
3. what algorithm is to be applied to the selected data to determine O3 and O1′.

1 Parameter(s) to be used

The parameters to be used must be selected. The magnetic field will be frequently used, due to its physical importance and the robustness of its measurement. But there are other possibilities: the plasma convection velocity, the electric field, and even the number density may be used in conjunction with at least one vector quantity.

2 Selection of the data interval

A - Time interval selection

Three types of data selection may be identified :

1. Fixed time interval. O3 and O1′ are determined by some sort of local "average". The GUI must allow specification of
   - the window within which data will be used to determine O3 and O1′.
2. Moving time interval. The principle is basically the same as before, except that the window is not fixed, it moves in time as the "target time" advances. The moving time interval allows a long sequence of data may be treated always using the "locally" determined directions O3 and O1′. The GUI must allow specification of :
   - the width of the window ;
   - how the window is situated with respect to the "target time" : for example,
     - centred symmetrically on the target time and advancing uniformly with time. This is the "sliding window" but, depending upon the algorithm used, it may be computationally heavy.
     - other solutions may be proposed, as suggested below.
3. Jump conditions. The directions O3 and O1′ are determined by some discrete event, such as the passage of a shock or a discontinuity. The distinguishing characteristic of this type of data selection is that O3 and O1′ are determined by two, possible non-contiguous, data intervals, one before and one after the event. The GUI must allow specification of :
   - two data intervals, one on each side of the event.

The application of a selection of type 2 to data containing significant discontinuities separated by characteristic time intervals which are neither much larger nor much smaller than the width of the window does not make scientific sense. On the contrary, a selection of type 1 can be applied to any data segment (even a discontinuity), while type 3 is reserved for discontinuities.

B - Implementation of the data selection

The following solutions are proposed :

1. Fixed time interval. This should be specified either
   - by entering the start time and stop time from the keyboard
   - by clicking on a graphical representation of some data which includes the interval of interest.
2. Moving time interval. It should be possible to enter via the keyboard :
   - the width of the data interval ;
   - the offset of the centre of the window with respect to the target time (normally zero) ;
   - the interval Δt at which this window is moved. All positive values of Δt are allowed.

     If $\Delta t = 0$, the window is always centred on the target time (this is the sliding window) ;

     If $\Delta t = T$, the window is moved by $2T$ only when the target time reaches some offset T from the centre of the window.

If $\Delta t = T =$ **window_half_width** (a special example of the preceding case), the window is moved by its own width only when the target time has moved by the characteristic half-width of the window.

3. This is the same as 1, but two intervals need to be specified.

3 - Algorithms to be used

There number of algorithms which may be used is unlimited, so these algorithms should be implemented as "plug-ins". We should start with a limited number of well-understood and well-documented algorithms, and add new algorithms as they become available.

The user will select the algorithm he wishes to use from a menu in the pop-up window. There will be algorithms applicable to a single window, and algorithms applicable to double window (jump conditions).

	Definition of O3	Definition of O1′
For a single time interval		
•	Mean magnetic field	The mean nadir
•	Mean magnetic field	The mean direction of the Sun
•	Mean satellite velocity	The mean magnetic field
•	Minimum magnetic variance	Maximum magnetic variance
For a pair of time intervals		
•	Magnetic field jump	Magnetic coplanarity
•	Rankine-Hugoniot jump conditions	Magnetic coplanarity
•	Rankine-Hugoniot jump conditions	Hoffman-Teller velocity

Note that it is further necessary to specify which physical variables are to be used with the Rankine-Hugoniot jump conditions, the possibilities include : the magnetic field, the density, the convection velocity, and probably others.

Coordinate System and acronym ↓		Direction of the ↓ axis defined by	Meridian containing the ↓ axis defined by
Motion whenever the spacecraft manoeuvres			
Despun Satellite	DS	O3 satellite spin axis	O1 direction of the Sun
Inverted Despun	IDS	O3 inverse of satellite spin axis	O1 direction of the Sun
Inertial systems			
Geocentric Equatorial Inertial	GEI	O3 Earth's North geographic pole	O1 first point of Aries (position of the Sun at the vernal equinox)
Ecliptic	ECL	O3 solar North ecliptic pole	O1 first point of Aries
Galactic	GAL	O3 Galactic pole	O1 Galactic centre
Annual motion			
Geocentric Solar Ecliptic	GSE	O3 solar North ecliptic pole	O1 direction of the Sun
Geocentric Solar Equatorial	GSEQ	O1 direction of the Sun	O2 Sun's equatorial plane
Annual & diurnal motions			
Geocentric Solar Magnetic	GSM	O1 direction of the Sun	O3 Earth's North magnetic pole
Solar Magnetic	SM	O3 Earth's North magnetic pole	O1 direction of the Sun
Diurnal motion			
Geographic	GEO	O3 Earth's North geographic pole	O1 Royal Greenwich Observatory
Geomagnetic	MAG	O3 Earth's North magnetic pole	O1 ?????
Local systems			
Dipole Meridian	DM	O3 Earth's North magnetic pole	O1 local zenith
Vertical Dusk Horizontal	VDH	O3 local zenith	O1 local eastwards direction
Data-dependent systems			
Magnetic field		O3 mean magnetic field	O1 see Section 3.4.1
Minimum variance		O3 eigenvector of minimum variance	O1 eigenvector of maximum variance
Boundary normal	LMN	O3 boundary normal as determined from the data	O1 direction of maximum variance

Table 2: Geophysical coordinate systems.

4 Cluster Specific Coordinate Transformations

Science data is measured in satellite coordinates, but should be presented in inertial coordinates for scientific analysis. The transformation from satellite coordinates to inertial coordinates involves a rapidly varying transformation (section 2.4). The data is obtained in sensor coordinates. The most widely-used inertial, or rather quasi-inertial, system for Cluster data analysis will probably be the GSE coordinate system. In any case, the transformation from one inertial or quasi-inertial system to another is relatively easy.

Therefore the passage from the sensor system to the GSE system will be split into three stages:

- sensor coordinates to the spin reference system
- spin reference to despun satellite system
- despun to an inertial system.

Further details are given in chapter 5.

4.1 SC → SR Coordinates

This section describes the four transformations required to pass successively from sensor (instrument) coordinates,

- to WEC coordinates (WEC),
- to mechanical build (BB),
- to attitude system (AS),
- to spin reference coordinates (SR).

4.1.1 SC → WEC Coordinates

Let the directions of the WEC search coil magnetic axes be $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$; these vectors will be close to, but not exactly the same as, the vectors $\mathbf{w}^i$. The measured components B_i^s of the $\mathbf{B}$-field in sensor coordinates are then

$$B_1^s = \mathbf{s}^1 \cdot \mathbf{B} \quad , \quad B_2^s = \mathbf{s}^2 \cdot \mathbf{B} \quad \text{and} \quad B_3^s = \mathbf{s}^3 \cdot \mathbf{B} \; . \tag{25}$$

Let the WEC coordinate axes be $\mathbf{w}^1$, $\mathbf{w}^2$ and $\mathbf{w}^3$ (*c.f.* Table 1). The three components B_i^w of the $\mathbf{B}$-field in mechanical build coordinates are given by analogous equations:

$$B_1^w = \mathbf{w}^1 \cdot \mathbf{B} \quad , \quad B_2^w = \mathbf{w}^2 \cdot \mathbf{B} \quad \text{and} \quad B_3^w = \mathbf{w}^3 \cdot \mathbf{B} \; . \tag{26}$$

It is shown in Appendix 1 (eqs. 58 and 60) that B_i^w may be expressed in terms of B_i^s by

$$B_i^w = R(ws)_{ij} B_j^s \tag{27}$$

where

$$R(ws)_{ij} = \frac{1}{2\,\det(\mathbf{s})}\; w_p^i \; \epsilon_{jkl} \; \epsilon_{pmn} \; s_m^k \; s_n^l \; . \tag{28}$$

where ϵ_{ijk} is the permutation operator. If WEC coordinates are used ($\mathbf{x}^i = \mathbf{w}^i$), and this expression simplifies to (*c.f.* eq. 3)

$$R(ws)_{ij} = \frac{1}{2\ \det(\mathbf{s})}\, \epsilon_{imn}\, \epsilon_{jkl}\, s^k_m\, s^l_n \tag{29}$$

with $\mathbf{s}^k_m = \mathbf{w}^m \cdot \mathbf{s}^k$. It is left as an exercise for the reader to demonstrate that these equations are equivalent to equations 2.3 and 2.4 of ref. 8 (eq. 29 is the "solution" of eq. 2.4 of ref. 8).

Equation 27 and either 28 or 29 can be translated into executable computer code.

Experimentally determined matrices $\mathbf{s}^k_m$ are presented in Appendix 3.

4.1.2 WEC → MB Coordinates

Equation 29 transforms to WEC coordinates. The transformation to mechanical build coordinates is

$$B^b_i = R(bw)_{ij}\ B^w_j \tag{30}$$

where, from eq. 4, the element i, j of the matrix $R(bw)$ is the projection of the vector $\mathbf{w}^j$ onto the vector $\mathbf{b}^i$. Thus, from eq. 14,

$$R(bw)_{ij} = \begin{pmatrix} R(bw)_{11} & R(bw)_{12} & R(bw)_{13} \\ R(bw)_{21} & R(bw)_{22} & R(bw)_{23} \\ R(bw)_{31} & R(bw)_{32} & R(bw)_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\ . \tag{31}$$

Note that eqs. 29 and 30 together are equivalent to eq. 28.

4.1.3 MB → AS coordinates

This coordinate change is special to Cluster (and a few other missions): it brings the ESTEC choice of satellite coordinates into line with the system used most frequently for spinning spacecraft, with Oz parallel to the mechanical build axis closest to the spacecraft spin axis.

For Cluster,

$$B^a_i = R(ab)_{ij} B^b_j \tag{32}$$

with,

$$R(ab)_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\ . \tag{33}$$

For ISEE and many other spacecraft this transformation reduces to a simple identity.

4.1.4 AS → SR coordinates

The spacecraft centre of mass expressed in satellite mechanical build coordinates can be found in the SATT file (see Sect. 3.1.5, Point 1, on page 9). Offsets from the true centre of mass from the origin of coordinates need to be taken into account only when the directions are required for objects near to the spacecraft; it is usually neglected in science data processing.

The spacecraft balance is generally good and the transformation from attitude to spin reference coordinates is small, but not always negligible. The Euler angles Ψ_1 and Ψ_2 can be found in the SATT file (see Sect. 3.1.5, Point 2, on page 10). In terms of these angles, the transformation can be written

$$B_i^r = R(ra)_{ij} B_j^a \tag{34}$$

with (ref. 3, page 137)

$$R(ra)_{ij} = \begin{pmatrix} \cos\Psi_2 & 0 & -\sin\Psi_2 \\ \sin\Psi_2 \ \sin\Psi_1 & \cos\Psi_1 & \sin\Psi_1 \ \cos\Psi_2 \\ \sin\Psi_2 \ \cos\Psi_1 & -\sin\Psi_1 & \cos\Psi_1 \ \cos\Psi_2 \end{pmatrix} . \tag{35}$$

4.2 SR → DS coordinates (despinning the data)

The coordinate transformations from spin reference to despun spacecraft coordinates has the particularity that the coordinate transformation matrix is varying as quickly as the spacecraft is spinning.

The transformation from spin reference to despun spacecraft coordinates is

$$B_i^d = R(dr)_{ij} B_j^r \tag{36}$$

with the rotation matrix obtained from the definitions of sections 3.1.5 and 3.1.6

$$R(dr)_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{37}$$

where ϕ is the spin phase, that is, the phase angle of the +YB axis of the spin reference (SR) system measured in the despun satellite (DS) system.

This definition of the spin phase is consistent with Annex 1 of ref. 3: on page 135 *(the text seems to have changed in Issue 3)* it is written that

- the phase of the spacecraft that Flight Dynamics Attitude Determination will provide is:
  - rotation angle of the half-plane defined by the maximum principal axis of inertia and the +YB axis, around the maximum principal axis of inertia from the time when the Sun direction was contained in this plane.

These two definitions are equivalent because the SR and the DS coordinate systems share the common O3 axis.

In reality, things are rather more complicated. The WEC understanding of the determination of the spin phase ϕ is supplied in Appendix 2.

4.3 DS → GEI coordinates

The base vectors of the DS coordinate are defined in terms of

$$\begin{aligned} \mathbf{x} &= \text{the direction of the spacecraft spin axis,} \\ \mathbf{h} &= \text{the direction of the Sun.} \end{aligned}$$

thus

$$\begin{aligned} \mathbf{d}^1 &= (\mathbf{h} - (\mathbf{x}\cdot\mathbf{h})\mathbf{x})/\alpha \\ \mathbf{d}^2 &= (\mathbf{x}\times\mathbf{h})/\alpha \\ \mathbf{d}^3 &= \mathbf{x} \end{aligned} \tag{38}$$

where

$$\alpha = \sqrt{1 - (\mathbf{x}\cdot\mathbf{h})^2} \tag{39}$$

Therefore the transformation from despun satellite to geocentric equatorial inertial coordinates is

$$B_i^e = R(ed)_{ij} B_j^d \tag{40}$$

with the rotation matrix

$$R(ed)_{ij} = \mathbf{e}^i \cdot \mathbf{d}^j \;=\; \frac{1}{\alpha}\begin{pmatrix} \mathbf{e}^1\cdot\mathbf{h} - (\mathbf{x}\cdot\mathbf{h})(\mathbf{e}^1\cdot\mathbf{x}) & [\mathbf{e}^1,\mathbf{x},\mathbf{h}] & \alpha\,\mathbf{e}^1\cdot\mathbf{x} \\ \mathbf{e}^2\cdot\mathbf{h} - (\mathbf{x}\cdot\mathbf{h})(\mathbf{e}^2\cdot\mathbf{x}) & [\mathbf{e}^2,\mathbf{x},\mathbf{h}] & \alpha\,\mathbf{e}^2\cdot\mathbf{x} \\ \mathbf{e}^3\cdot\mathbf{h} - (\mathbf{x}\cdot\mathbf{h})(\mathbf{e}^3\cdot\mathbf{x}) & [\mathbf{e}^3,\mathbf{x},\mathbf{h}] & \alpha\,\mathbf{e}^3\cdot\mathbf{x} \end{pmatrix}. \tag{41}$$

To evaluate the elements of this matrix, we dispose of the following information:

- The right ascension α_x and declination δ_x of the spin axis (Sect. 3.1.6, Point 3, page 10), in terms of which

$$\mathbf{e}^i\cdot\mathbf{x} \;=\; \begin{pmatrix} \cos\delta_x\cos\alpha_x \\ \cos\delta_x\sin\alpha_x \\ \sin\delta_x \end{pmatrix} \tag{42}$$

- The position $\mathbf{h}$ of the Sun in ecliptic (ECL) coordinates,

$$h_i^h = (\cos\phi_h,\ \sin\phi_h,\ 0)\ ,$$

  where the ecliptic longitude ϕ_h of the Sun may be computed (Sect. 3.3.1, page 12). $\mathbf{h}$ may be expressed in GEI coordinates via eq. 17, thus

$$\mathbf{e}^i\cdot\mathbf{h} = R(eh)_{ij}\,h_j^h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\delta_p & \cos\delta_p \\ 0 & -\cos\delta_p & \sin\delta_p \end{pmatrix}\begin{pmatrix} \cos\phi_h \\ \sin\phi_h \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi_h \\ \sin\phi_h\sin\delta_p \\ -\sin\phi_h\cos\delta_p \end{pmatrix}. \tag{43}$$

Furthermore, we note that

$$\begin{aligned} \mathbf{x}\cdot\mathbf{h} &= \sum_{i=1}^{3}\left(\mathbf{e}^i\cdot\mathbf{x}\right)\left(\mathbf{e}^i\cdot\mathbf{h}\right) \\ &= \cos\delta_x\cos\alpha_x\cos\phi_h + \cos\delta_x\sin\alpha_x\sin\phi_h\sin\delta_p - \sin\delta_x\sin\phi_h\cos\delta_p \end{aligned} \tag{44}$$

and that

$$\begin{aligned} [\mathbf{e}^i,\mathbf{x},\mathbf{h}] &= \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\left(\mathbf{e}^j\cdot\mathbf{x}\right)\left(\mathbf{e}^k\cdot\mathbf{h}\right) \\ &= \begin{pmatrix} -(\cos\delta_x\sin\alpha_x\sin\phi_h\cos\delta_p + \sin\delta_x\sin\phi_h\sin\delta_p) \\ \sin\delta_x\cos\phi_h + \sin\phi_h\cos\delta_p\cos\delta_x\cos\alpha_x \\ \cos\delta_x\cos\alpha_x\sin\phi_h\sin\delta_p - \sin\phi_h\sin\delta_p\cos\phi_h \end{pmatrix}. \end{aligned} \tag{45}$$

Thus all the elements of the rotation matrix $R(ed)_{ij}$ of eq. 41 may be evaluated.

4.4 GEI → GSE coordinates

The transformation from GEI to GSE coordinates is

$$B_i^g = R(ge)_{ij} B_j^e$$

where the rotation matrix $R(ge)_{ij}$ may be expressed in terms of the rotations represented by the matrices of eqs. 20 and 17,

$$R(ge)_{ij} = R(gh)_{ik} R(he)_{kj} = \begin{pmatrix} \cos\phi_h & \sin\phi_h \sin\delta_h & -\sin\phi_h \cos\delta_h \\ -\sin\phi_h & \cos\phi_h \sin\delta_h & -\cos\phi_h \cos\delta_h \\ 0 & \cos\delta_h & \sin\delta_h \end{pmatrix} \quad (46)$$

4.5 DS → GSE coordinates

Since the origin of spin phase is the meridian plane which contains the direction of the Sun, for a spacecraft oriented with its spin axis close to the north ecliptic pole this transformation is close to, but nevertheless significantly different from, unity. For other spacecraft, such as Ulysses, this transformation may represent a major rotation of axes. The case of a spacecraft, such as Cluster, with its spin axis close to the south ecliptic pole will be described in Sect. 4.6.2.

The transformation from despun to geocentric solar ecliptic coordinates is (eq. 7)

$$B_i^g = R(gd)_{ij} B_j^d$$

where

$$R(gd)_{ij} = R(ge)_{ik} R(ed)_{kj} \ . \quad (47)$$

and $R(ge)_{ik}$ and $R(ed)_{kj}$ are determined respectively by eqs. 46 and 41 through 45. But $R(gd)_{ij}$ may also be derived from first principles. The despun satellite and the geocentric solar ecliptic coordinate systems are together defined by only three vectors:

$$\begin{aligned} \mathbf{x} &= \text{the direction of the spacecraft spin axis,} \\ \mathbf{p} &= \text{the direction of the ecliptic north pole,} \\ \mathbf{h} &= \text{the direction of the Sun.} \end{aligned}$$

In terms of these vectors, the base vectors of the despun satellite and geocentric solar coordinate systems are respectively

$$\left.\begin{array}{lllclll} \mathbf{d}^1 &=& (\mathbf{h} - (\mathbf{x}\cdot\mathbf{h})\mathbf{x})/\alpha & & \mathbf{g}^1 &=& (\mathbf{h} - (\mathbf{p}\cdot\mathbf{h})\mathbf{p})/\beta \\ \mathbf{d}^2 &=& (\mathbf{x}\times\mathbf{h})/\alpha & & \mathbf{g}^2 &=& (\mathbf{p}\times\mathbf{h})/\beta \\ \mathbf{d}^3 &=& \mathbf{x} & \text{and} & \mathbf{g}^3 &=& \mathbf{p} \\ \alpha &=& +\sqrt{1-(\mathbf{x}\cdot\mathbf{h})^2} & & \beta &=& +\sqrt{1-(\mathbf{p}\cdot\mathbf{h})^2} \end{array}\right\} \quad (48)$$

Since both sets of base vectors ($\mathbf{d}^i$ and $\mathbf{g}^i$) are orthonormal, the required transformation matrix $R(gd)$ of eq. 47 is given by eq. 4,

$$R(gd)_{ij} = \mathbf{g}^i.\mathbf{d}^j = \frac{1}{\alpha\beta} \times$$

$$\begin{pmatrix} 1-(\mathbf{x}.\mathbf{h})^2-(\mathbf{p}.\mathbf{h})^2+(\mathbf{x}.\mathbf{p})(\mathbf{x}.\mathbf{h})(\mathbf{p}.\mathbf{h}) & (\mathbf{p}.\mathbf{h})[\mathbf{x},\mathbf{p},\mathbf{h}] & \alpha\{(\mathbf{x}.\mathbf{h})-(\mathbf{p}.\mathbf{h})(\mathbf{x}.\mathbf{p})\} \\ -(\mathbf{x}.\mathbf{h})[\mathbf{x},\mathbf{p},\mathbf{h}] & (\mathbf{x}.\mathbf{p})-(\mathbf{p}.\mathbf{h})(\mathbf{x}.\mathbf{h}) & \alpha[\mathbf{x},\mathbf{p},\mathbf{h}] \\ \beta\{(\mathbf{p}.\mathbf{h})-(\mathbf{x}.\mathbf{h})(\mathbf{x}.\mathbf{p})\} & -\beta[\mathbf{x},\mathbf{p},\mathbf{h}] & \alpha\beta(\mathbf{x}.\mathbf{p}) \end{pmatrix}$$

where $[\mathbf{x}, \mathbf{p}, \mathbf{h}]$ is the triple scalar product $\mathbf{x} \cdot (\mathbf{p} \times \mathbf{h})$. Since $(\mathbf{p}.\mathbf{h}) = 0$, this simplifies to

$$R(gd)_{ij} = \frac{1}{\alpha} \times \begin{pmatrix} \alpha^2 & 0 & \alpha(\mathbf{x}\cdot\mathbf{h}) \\ -(\mathbf{x}\cdot\mathbf{h})[\mathbf{x},\mathbf{p},\mathbf{h}] & (\mathbf{x}\cdot\mathbf{p}) & \alpha[\mathbf{x},\mathbf{p},\mathbf{h}] \\ -(\mathbf{x}\cdot\mathbf{h})(\mathbf{x}\cdot\mathbf{p}) & -[\mathbf{x},\mathbf{p},\mathbf{h}] & \alpha(\mathbf{x}\cdot\mathbf{p}) \end{pmatrix} \tag{49}$$

To evaluate the scalar products, we note that the directions of these three vectors are available, as follows:

- the right ascension α_x and declination δ_x (*i.e.*, the GEI system) of the spin axis $\mathbf{x}$, Sect. 3.1.5, Point 3, page 10 ;
- the ecliptic pole $\mathbf{p}$ is the O3 axis of the ECL system (Sect. 3.2.2, page 11) ;
- the direction $\mathbf{h}$ of the Sun is expressed simply in ECL coordinates in terms of its ecliptic longitude ϕ_h (Sect. 3.3.1, page 12).

Therefore we evaluate the scalar products in ecliptic (ECL) coordinates, the direction cosines then being (using eq. 17 to transform $\mathbf{x}$),

$$\mathbf{x} = \begin{pmatrix} \cos\delta_x \cos\alpha_x \\ \cos\delta_p \sin\delta_x + \sin\delta_p \cos\delta_x \sin\alpha_x \\ \sin\delta_p \sin\delta_x - \cos\delta_p \cos\delta_x \sin\alpha_x \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} \cos\phi_h \\ \sin\phi_h \\ 0 \end{pmatrix}$$

It is straightforward to evaluate the scalar products occurring in eq. 49 :

$$\left.\begin{aligned} (\mathbf{x}\cdot\mathbf{p}) &= \sin\delta_p \sin\delta_x - \cos\delta_p \cos\delta_x \sin\alpha_x \\ (\mathbf{x}\cdot\mathbf{h}) &= \cos\phi_h \cos\delta_x \cos\alpha_x + \sin\phi_h (\sin\delta_p \cos\delta_x \sin\alpha_x + \cos\delta_p \sin\delta_x) \\ [\mathbf{x},\mathbf{p},\mathbf{h}] &= \cos\phi_h (\cos\delta_p \sin\delta_x + \sin\delta_p \cos\delta_x \sin\alpha_x) - \sin\phi_h \cos\delta_x \cos\alpha_x \end{aligned}\right\} \tag{50}$$

In Appendix 4 formally identical expressions are derived using GEI coordinates. *These expressions have not yet been compared with those given, for example, in ref. 8, pages 29-30.*

Spin-stabilised spacecraft often have their spin axis pointing towards one or other of the ecliptic poles ; this reduces thermal and electrical power supply problems. Furthermore, the spin axis is often tilted slightly towards or away from the Sun, to avoid boom-mounted sensors passing into eclipse behind the satellite as it rotates. For a spacecraft with its spin axis $\mathbf{x}$ tilted away from the north ecliptic pole $\mathbf{p}$ though an angle $\theta_{\mathrm{N}} < \pi/2$ towards the Sun $\mathbf{h}$, that is, with $\mathbf{x}$ in the plane of $\mathbf{p}$ and $\mathbf{h}$ so that $[\mathbf{x}, \mathbf{p}, \mathbf{h}] = 0$, eqs. 50 may be written $(\mathbf{x} \cdot \mathbf{p}) = \alpha = \cos\theta_{\mathrm{N}}$, $(\mathbf{x} \cdot \mathbf{h}) = \sin\theta_{\mathrm{N}}$, and eq. 49 becomes

$$R(gd)_{ij} = \begin{pmatrix} (\mathbf{x}\cdot\mathbf{p}) & 0 & (\mathbf{x}\cdot\mathbf{h}) \\ 0 & 1 & 0 \\ -(\mathbf{x}\cdot\mathbf{h}) & 0 & (\mathbf{x}\cdot\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{N}} & 0 & \sin\theta_{\mathrm{N}} \\ 0 & 1 & 0 \\ -\sin\theta_{\mathrm{N}} & 0 & \cos\theta_{\mathrm{N}} \end{pmatrix} \tag{51}$$

When $\cos\theta_{\mathrm{N}}$ is small, the DS system is close to the GSE system. Nevertheless, to determine $R(gd)_{ij}$ precisely either eq. 47 or eqs. 49 and 50 must be used.

4.6 IDS coordinates

For a spacecraft which has its spin axis close to the south ecliptic pole, the DS and GSE systems are quite different. As described in Sect. 3.1.7, for such spacecraft the Inverted Despun (IDS) coordinate system is introduced because it is close to GSE while still preserving the relative simplicity of the DS system. Of course, it is not necessary to use the IDS system ; if the user wants data in the GSE system, he should transform directly from the DS to the GSE system using eqs. 49 and 50. Eq. 52 is provided in case that the IDS system is considered useful, and eq. 53 is provided for completeness.

4.6.1 DS → IDS coordinates

As explained in Sect. 3.1.7, this transformation is simply a rotation through 180° about the O1-axis

$$R(id)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (52)$$

4.6.2 IDS → GSE coordinates

This transformation may be derived from eqs. 49 and 52, using $R(id)^{-1}_{jk} = R(id)_{jk}$:

$$R(gi)_{ij} \quad = \quad R(gd)_{ij}\, R(di)^{-1}_{jk} \quad = \quad \frac{1}{\alpha} \times \begin{pmatrix} \alpha^2 & 0 & -\alpha(\mathbf{x}\cdot\mathbf{h}) \\ -(\mathbf{x}\cdot\mathbf{h})[\mathbf{x},\mathbf{p},\mathbf{h}] & -(\mathbf{x}\cdot\mathbf{p}) & -\alpha[\mathbf{x},\mathbf{p},\mathbf{h}] \\ -(\mathbf{x}\cdot\mathbf{h})(\mathbf{x}\cdot\mathbf{p}) & [\mathbf{x},\mathbf{p},\mathbf{h}] & -\alpha(\mathbf{x}\cdot\mathbf{p}) \end{pmatrix} \quad (53)$$

For a spacecraft with its spin axis $\mathbf{x}$ in the plane of $\mathbf{p}$ and $\mathbf{h}$ and tilted away from the *south* ecliptic pole $-\mathbf{p}$ though an angle $\theta_S < \pi/2$, $[\mathbf{x},\mathbf{p},\mathbf{h}] = 0$ and eqs. 50 may be written $-(\mathbf{x}\cdot\mathbf{p}) = \alpha = +\cos\theta_S$ and $(\mathbf{x}\cdot\mathbf{h}) = \sin\theta_S$, so that eq. 53 becomes

$$R(gi)_{ij} \quad = \quad \begin{pmatrix} -(\mathbf{x}\cdot\mathbf{p}) & 0 & -(\mathbf{x}\cdot\mathbf{h}) \\ 0 & 1 & 0 \\ (\mathbf{x}\cdot\mathbf{h}) & 0 & -(\mathbf{x}\cdot\mathbf{p}) \end{pmatrix} \quad = \quad \begin{pmatrix} \cos\theta_S & 0 & -\sin\theta_S \\ 0 & 1 & 0 \\ \sin\theta_S & 0 & \cos\theta_S \end{pmatrix}. \quad (54)$$

To determine $R(gi)_{ij}$ precisely, eqs. 50 and 53 must be used.

4.6.3 Pedagogical Digression

The expressions 51 and 54 differ in the sense of their rotation ; this is as expected, because the "tilts" of the spin axes are in opposite directions. Nevertheless, eq. 54 cannot be derived simply by putting $\theta_S = \pi - \theta_N$ in eq. 51. Indeed, as $\theta_N \to \pi/2$ and $\theta_S \to \pi/2$, the expressions 51 and 54 tend respectively to $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, despite the fact that in both cases the spin axis tends to the same direction. This difference is due to the fact that when the spin axis $\mathbf{x}$ is in the direction $\mathbf{h}$ of the Sun $\alpha = 0$ and the $\mathbf{d}^1$ and $\mathbf{d}^2$ axes (see eq. 48), and thus the DS system, cannot be defined. When $\mathbf{x}$ changes direction while remaining in the plane of $\mathbf{p}$ and $\mathbf{h}$, as θ passes the value $\pi/2$ the DS coordinate system suffers a discontinuous rotation or "jump" of 180° about its 03-axis.

We now consider what happens when $\mathbf{x}$ varies in a meridian (of $\mathbf{p}$) which is perpendicular to the meridian containing $\mathbf{h}$. Let $\mathbf{x}$ be tilted though an angle θ_N from the north celestial pole ; then eqs. 50 simplify to $\alpha = 1$, $(\mathbf{x}\cdot\mathbf{h}) = 0$, $(\mathbf{x}\cdot\mathbf{p}) = \cos\theta_N$, and $[\mathbf{x},\mathbf{p},\mathbf{h}] = \pm\sin\theta_N$ with the sign depending on which of two meridian planes is chosen. On the other hand, if $\mathbf{x}$ is tilted in the same meridian though an angle θ_S from the south celestial pole, eqs. 50 simplify to $\alpha = 1$,

$(\mathbf{x}\cdot\mathbf{h}) = 0$, $(\mathbf{x}\cdot\mathbf{p}) = -\cos\theta_{\rm S}$, and $[\mathbf{x},\mathbf{p},\mathbf{h}] = \pm\sin\theta_{\rm S}$. For these two cases eqs. 49 and 53 become respectively

$$R(gd)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{\rm N} & \pm\sin\theta_{\rm N} \\ 0 & \mp\sin\theta_{\rm N} & \cos\theta_{\rm N} \end{pmatrix} \quad \text{and} \quad R(gi)_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{\rm S} & \mp\sin\theta_{\rm S} \\ 0 & \pm\sin\theta_{\rm S} & \cos\theta_{\rm S} \end{pmatrix}$$

As in the case of the expressions 51 and 54, the two rotations are of opposite sense because the two “tilts” of the spin axes are in opposite directions. Furthermore, these two rotation matrices can be derived one from the other by using $\theta_{\rm S} = \pi - \theta_{\rm N}$ because, in the meridian orthogonal to **h**, there is no discontinuity when $\theta = \pi/2$.

Finally, after examination of the matrices $R(gd)_{ij}$ and $R(gi)_{ij}$ of eqs. 49 and 53 when the spin axis **x** lies firstly in the meridian containing **h** then, secondly, in the perpendicular meridian, we conclude that :

- $R(gd)_{ij}$ and $R(gi)_{ij}$ exhibit the correct behaviour in these two special cases,
- the full expressions of eq. 49 or 53 should always be used to transform to GSE coordinates, and
- the DS or DSI systems as defined in Sections 3.1.6 and 3.1.7 should not be used when the spin axis **x** is close to the direction of the Sun.

This latter situation does occur, for example, for Ulysses. Then some direction other than that of the Sun must be used to define the vector **h**.

5 Implementation Issues

In chapters 3 and 4 total of 17 different coordinate systems were defined, which leads to 136 different coordinate transformations. WEC/ISDAT data files are of finite length, and their structure requires that with each data file be associated the corresponding meta-data, as described in ref. 6.

In order to simplify the handling of the meta-data, it is desirable that all coordinate transformations be performed by one common coordinate transformation operator. To simplify the task of producing such a general coordinate transformation operator, the relationships between these different coordinate systems must be structured.

The numerical algorithms of ref. 5 will be used for the general transformations. It is important to note, however, that ref. 5 deals with coordinate systems on behalf of the numerical simulation network, which has requirements which are more simple than those for the analysis of real data. Numerical simulation takes a "snapshot" view of its numerical model; the data being analysed is generally acquired during a finite time interval, during which the coordinate transformations will generally be evolving. The methodology of ref. 5 will be modified to take this into account.

5.1 Families of Coordinate Systems

Coordinate transformations systems can be distinguished between those which are time-varying and those which are constant. This distinction can be used to separate the coordinate systems of chapter 3 into several groups or families; within each family the internal coordinate transformations are non time-varying.

Fig. 1 shows all the coordinate systems of chapter 3 grouped together in this way. Between each family and the GEI inertial coordinate system there is a time-varying transformation; in Fig. 1 these transformations are written in *italics*.

Each time-varying transformation is defined between the first, or primary, member of the family and the GEI (inertial) coordinate system; when transformation to some other (non-primary) system is required, this is done by transformation to the primary system of that group, then transformation to the desired system; this latter transformation is relatively simple, because it is constant (non time-varying).

5.2 Categories of Coordinate Transformation

Several types of coordinate transformation have been identified in section 2.4.

From the point of view of implementation, there are really only three categories to consider:

Non-varying transformations, for which time variation may be totally neglected. Transformations between inertial systems vary slowly enough for them to be considered constant over time intervals short compared with one year. The transformation matrix is defined by a standard data structure described in section 2.2, with only one data structure valid throughout the whole duration of the time interval being studied. (In reality, even these transformations are not strictly constant; hence the difference between epoch 1950 and epoch 2000 GEI coordinates.)

Time varying rotations, for which the rotation matrix may be derived by linear joining of a logical instrument representing the coordinate transformation.

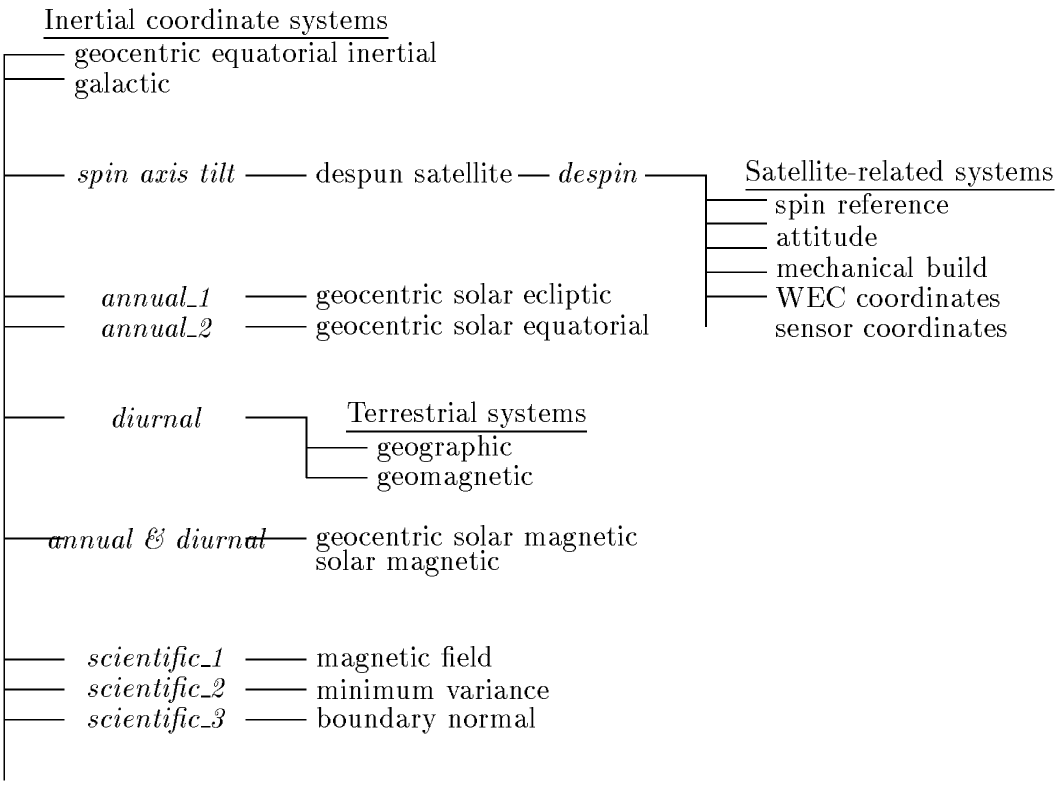

Figure 1: Hierarchy of coordinate systems showing (in *italics*) time-varying coordinate transformations. Coordinate transformations which do not vary in time are not explicitly mentioned.

The joining algorithm could be the standard algorithm, although a private private "despin" linear joining algorithm would have its advantages. Assuming that the joining algorithm is linear, then the logical instrument describing the coordinate transformation must be generated with adequate time resolution. It was shown in section 2.2 that the following resolutions are adequate:

- 1 day for transformations with an annual variation;
- 4 mn for transformation with a daily variation.

Despin, for which the transformation matrix is varying so rapidly that it is more reasonable to join the spin phase rather than the transformation matrix itself.

Therefore a "spin phase" logical instrument must be generated with adequate time resolution. The spin rate varies so slowly that the spin phase can be joined linearly within time intervals extending over several hundreds of rotations provided, of course, that there are no jumps of 2π within the interval.

5.3 The Transformation Procedure

To determine how to proceed with any particular coordinate transformation, Fig. 1 is examined. The coordinate systems either side of a time-varying coordinate transformation (in *italics*) are called the "primary" coordinate systems. The passage from any one coordinate system to another involves one or more rotations.

The transformation from any other of the coordinate systems of Fig. 1 to the nearest basic coordinate system is a non-varying transformation. These non-varying transformations are performed either by passing sequentially via the listed intermediate systems, or directly to or from the appropriate basic coordinate system.

Whether the coordinate transformation be non-varying, varying or despin, there are a certain number of common operations which must always be performed. The different operations are listed in Table 3, which also shows for each operation the categories of transformation for which it must be applied.

Note that the operations performed at steps 11.1.1, 11.2.1 and 11.2.3 are identical, and may be executed by the same piece of code.

5.4 "Transformation" Logical Instruments

It is clear that associated with the generalised coordinate transformation operator there will be a number of logical instruments describing the various elementary transformations required to achieve the overall transformation. These logical instruments will span the whole time interval of interest, and will contain individual data objects uniformly spaced in time with resolution adequate, namely

- 1 day for annually varying rotations,
- 4mn for daily rotations.

In addition, to permit joining, every logical instrument will include at least two time stamped data objects, evaluated at the start and the end of the data interval. In particular, this requirement extends to

- non varying transformations and
- logical instruments spanning less than one day of an annually varying transformation.

These transformation logical instruments are re-usable, and therefore they may be kept for subsequent use (TBC). In this case, they must be clearly labelled with some naming convention so that they can be identified for future use.

5.5 The "Spin Phase" Logical Instrument

Despin is coordinate transformation using a transformation matrix which is changing so rapidly with time that joining must be performed on the spin phase rather than the transformation matrix itself.

The rotation matrix is given in terms of the spin phase by eq. 37 of section 4.2. The computation of the rotation matrix from the spin phase is performed by the common coordinate transformation operator.

The spin phase ϕ is rapidly changing and must be re-computed for every time-stamp of the input data object. It is the responsibility of the general rotation operator to ensure that the spin phase is correctly joined to the data of the logical instrument being processed. Exactly how this is done has not yet been defined: it could either be via a "spin phase" logical instrument plus use of the general joining operator (see ref. 7), or via a spin phase function which returns the spin phase evaluated at the time supplied as an argument on entry.

Step	Transformations involving despin ↓ Time-varying transformations ↓ Non varying transformations ↓			
1.	Compare the meta-data of the input data object and the transformation matrix to verify that the attributes (rank, coordinate system, etc.) of the input are compatible with the transformation demanded. If the input data object is a two-dimensional, then only despin can be performed. If despin is required to perform the transformation, verify that the appropriate "spin phase" logical instrument is available. If any incompatibilities are found, an "informative error message" should be returned	x	x	x
2.	Determine the sequence of intermediate coordinate transformations using the information of Table 1	x	x	x
3.	Stamp the output data object with the appropriate meta-data; the essential minimum (and perhaps the only) change required is to change the reference "known WEC coordinate system"	x	x	x
4.	Determine whether the transformation is non-varying, slowly varying or rapidly varying.	x	x	x
5.	Accept the standard WEC/ISDAT structure describing the transformation matrix and its meta-data.	x		
6.	Accept all standard WEC/ISDAT structures describing the transformations BEFORE the despin; concatenate them (multiply them together).		x	x
7.	Accept all standard WEC/ISDAT structures describing the transformations AFTER the despin; concatenate them.		x	x
8.	Compute the rotation matrix corresponding to the centre of the time interval spanned by the data in the file.		x	
9.	Concatenate the rotations BEFORE despin, the despin matrix at the centre of interval, and the rotations AFTER despin.		x	
10.	FOR each data structure (*i.e.*, each time) of the input data file			
11.1	IF not rapidly varying			
11.1.1	Apply appropriate (vector or tensor) transformation, dimension by dimension of the data structure.	x	x	
11.2	ELSE			
11.2.1	Apply matrix representing concatenation of transformations BEFORE despin			x
11.2.2	Apply DESPIN algorithm			x
11.2.3	Apply matrix representing concatenation of transformations AFTER despin			x
11.2.4	ENDIF			
12.	REPEAT from step 10 for all the time-stamped structures of the input data object.	x	x	x
13.	Compute execution info/error flag (as necessary) and return.	x	x	x

Table 3: Step-by-step implementation of coordinate transformation.

If the general purpose joining operator is used, then the spin phase must increase monotonically throughout the time interval spanned by the spin phase logical instrument data object. Indeed, it is advisable that this be imposed as **a general requirement on the spin phase logical instrument** in order to avoid having to define a special "spin-joining" operator. This is important: the spin phase does indeed increases monotonically with time, but the value generally quoted is modulo 2π, which would create havoc if supplied to the standard joining algorithm.

5.6 Two-dimensional dimensional vectors

It has already been remarked that despin is the only coordinate transformation likely to be applied to two-dimensional vectors. As the general transformation operator can handle despin, is must also be capable of handling two-dimensional vectors. *Nevertheless, further study is required.*

6 Reference Documents

1. Experiment Interface Document, Part A, ESTEC document CL–EST–RS–0002/EID A, Issue 2, Rev. 0, 1993/06/01.
2. Experiment Interface Document, Part B, ESTEC document CL–EST–RS–0451/EID B, Issue 2, Rev 0, 1994/06/30.
3. Data Delivery Interface Document (DDID), ESOC document CL–ESC–ID–2001, Issue 3.0, 2000/05/19.
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5. Coordinate Transformation Library, by P. Robert, CRPE document DT/CRPE/1231, 1993/07.
6. The structure of the WEC/ISDAT data, WEC/ISDAT document CWD–OBSPM–DD–001.
7. Requirements for the Joining of Cluster WEC Data, WEC/ISDAT document CWD–OBSPM–TN–001, issue 1, rev. 2, dated 1994/09/26.
8. Fluxgate magnetometer data processing for Cluster, FGM-DAPCLUS, Data Processing Handbook, CL–IGM–SN–0001, Braunschweig, 1993/06/08.
9. ISDAT 2.0 Definition of Filter, WEC/ISDAT document CWD-IRF-TN-001.
10. Geophysical Coordinate Transformations, by C.T. Russell, *Cosmic Electrodyn.*, **2**, 184–196, 1971.
11. Space Physics Coordinate Transformations, by M.A. Hapgood, *Planet. Space Sci.*, **40**, 711–717, 1992.
12. Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, HMSO, London, 1961.
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APPENDIX 1

Transformation from a Non-orthogonal Coordinate System

If $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are the three sensor axes, let us define

$$\mathbf{s}'^1 = \mathbf{s}^2 \times \mathbf{s}^3 \,, \quad \mathbf{s}'^2 = \mathbf{s}^3 \times \mathbf{s}^1 \,, \quad \mathbf{s}'^3 = \mathbf{s}^1 \times \mathbf{s}^2 \,. \tag{55}$$

The vectors $\mathbf{s}'^\beta$ and $\mathbf{s}^\beta$ are the covariant and contravariant base vectors of the non-orthogonal sensor coordinate system. If the sensor axes were orthogonal, we would have $\mathbf{s}'^\beta = \mathbf{s}^\beta$. It easy to show that they satisfy

$$\mathbf{s}^\beta \cdot \mathbf{s}'^\mu = \det(\mathbf{s})\, \delta_{\beta\mu} \tag{56}$$

where $\det(\mathbf{s})$ is the determinant of the matrix $\mathbf{s}^\beta$ (or of the matrix $\mathbf{s}'^\beta$, the two determinants being equal). Furthermore, the expression

$$\sum_{\beta=1}^{3} \mathbf{s}'^\beta \mathbf{s}^\beta$$

represents an operator. Let us examine the effect of it operating on the base vector $\mathbf{s}^\mu$; using eq. 56,

$$\mathbf{s}^\mu \sum_{\beta=1}^{3} \mathbf{s}'^\beta \mathbf{s}^\beta = \sum_{\beta=1}^{3} \mathbf{s}^\mu \mathbf{s}'^\beta \mathbf{s}^\beta = \sum_{\beta=1}^{3} \det(\mathbf{s})\, \delta_{\beta\mu}\, \mathbf{s}^\beta = \det(\mathbf{s})\, \mathbf{s}^\mu \,.$$

Since this is true for $\mu = 1$, 2 or 3, it is also true for any linear combination of these three vectors (*i.e.*, for any vector); therefore this operator is a multiple of the identity operator $\mathcal{I}$:

$$\sum_{\beta=1}^{3} \mathbf{s}'^\beta \mathbf{s}^\beta = \det(\mathbf{s})\, \mathcal{I} \,. \tag{57}$$

Eqs. 56 and 57 are the generalisation to non-orthogonal coordinate systems of eqs. 3 and 6.

Let the WEC coordinate axes be $\mathbf{w}^1$, $\mathbf{w}^2$ and $\mathbf{w}^3$. The three components B_i^w of the $\mathbf{B}$-field in WEC coordinates are given by eqs. 26. We may use eq. 57 to express B_i^w in terms of B_i^s,

$$B_i^w = \mathbf{w}^i \cdot \mathbf{B} = \frac{1}{\det(\mathbf{s})}\, \mathbf{w}^i \cdot \sum_{\beta=1}^{3} \mathbf{s}'^\beta \mathbf{s}^\beta \cdot \mathbf{B} = \frac{1}{\det(\mathbf{s})} \sum_{\beta=1}^{3} (\mathbf{w}^i \cdot \mathbf{s}'^\beta)(\mathbf{s}^\beta \cdot \mathbf{B})$$

which may be written

$$B_i^w = R_{ij} B_j^s \tag{58}$$

where

$$R_{ij} = \frac{1}{\det(\mathbf{s})}\, \mathbf{w}^i \cdot \mathbf{s}'^j = \frac{1}{\det(\mathbf{s})}\, \mathbf{w}^i \cdot (\mathbf{s}^k \times \mathbf{s}^l) \quad \text{with } j, k, l \text{ cyclic} \,. \tag{59}$$

This expression is formal: to actually evaluate R_{ij} the vectors $\mathbf{w}^i$, $\mathbf{s}^k$ and $\mathbf{s}^l$ must be represented in some coordinate system. In an arbitrary coordinate system defined by base vectors $\mathbf{x}^i$ which are mutually orthogonal (hence $\mathbf{x}^i \neq \mathbf{s}^i$), this is

$$\begin{aligned} R_{ij} &= \frac{1}{\det(\mathbf{s})}\, w_p^i\, \epsilon_{pmn}\, s_m^k\, s_n^l \quad \text{with } j, k, l \text{ cyclic} \\ &= \frac{1}{2\,\det(\mathbf{s})}\, w_p^i\, \epsilon_{jkl}\, \epsilon_{pmn}\, s_m^k\, s_n^l \,. \end{aligned} \tag{60}$$

where ϵ_{ijk} is the permutation operator, $w_p^i = \mathbf{x}^p \cdot \mathbf{w}^i$, and $s_m^k = \mathbf{x}^m \cdot \mathbf{w}^k$.

APPENDIX 2

Determination of the Cluster Spin Phase

The following text from Per-Arne Lindqvist (corrected for the numerical error pointed out by Simon Walker) describes how the Cluster spin phase is to be determined from the data on the Cluster CD-ROM.

```
From: PLAFYS::MX%"lindqvist@plasma.kth.se" 24-NOV-1994 10:06:49.82
To: MX%"wec.data@irfu.irfu.se"
CC: MX%"cornilleau@rpevzb.cetp.ipsl.fr",MX%"pdecreau@cnrs-orleans.fr",
 MX%"gg@irfu.irfu.se",MX%"torkar@fiwf01.dnet.tu-graz.ac.at",
 MX%"nilsson@plasma.kth.se",MX%"mehra@plasma.kth.se",
 MX%"olsson@plasma.kth.se",MX%"blomberg@plasma.kth.se"
Subj: Cluster spin phase



Dear Colleagues,

This message should close an action item from the WEC meeting in Meudon
15-16 September 1994 (action item 32 in e-mail from Nicole) and an
action item from the WEC data meeting in Uppsala 19-20 September (noted
by myself, but not in the minutes?), to provide information on how to
obtain the Cluster spin phase from the raw data. It may also help Chris
to close AI 94-2.18 from the WEC data meeting.

-------------------------------------------------------------------------

Note on Cluster spin phase information.
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Per-Arne Lindqvist
24 November 1994

This note intends to summarize my current understanding of how to
extract the Cluster spin phase from the raw data on the CD-ROM.

The instantaneous spin phase information is available in the Satellite
housekeeping data in the form of Sun Reference Pulses (SRPs). How to
extract this information is, in principle, described in the answer from
ESOC to Anomaly Report (AR) 38 on the Data Delivery Interface Document
(DDID) version 2.3. There are, however, some inconsistencies in bit and
byte numbering, so I try to describe the procedure as follows.

Look in the Housekeeping Parameter Definition (HPD) file in the CD-ROM
to find the quantity described as "MER SUN EVENT". There are 12 such
quantities. The names of the HPD files (on the test CD-ROM), and the
mnemonics for the 12 quantities are:

Satellite File nameMnemonic in HPDByte offset

Cluster1 940901sd.1a11M_250, 251 ... 261234, 237, ... 267
Cluster2 940901sd.1a22M_250, 251 ... 261234, 237, ... 267
```

```
Cluster3 940901sd.1a33M_250, 251 ... 261234, 237, ... 267
Cluster4 940901sd.1a44M_250, 251 ... 261234, 237, ... 267
```

```
For each quantity, get the Byte offset and the Bit offset. Note that the
information in the HPD file of the test CD-ROM is incorrect. The byte
offset should be increased by 4, as stated in AR 6 on DDID 2.3. This
error in the byte offset is present in all HPD files for all instruments
on the test CD-ROM, and will be corrected in future CD-ROMs. The table
above contains the correct offsets.

According to the HPD file, the bit offset is 3 for all these quantities,
which is correct, using the bit numbering as specified on page 96 of
DDID 2.3 (MSB = bit 0, LSB = bit 7). To get the SRP, test this bit (bit
3) of the above stated quantities. When the bit is 1, we have an EVENT
(as correctly stated in the CALINF part of the HPD file information).
For each such event, extract bits 4-23 of the quantity (remember the bit
numbering convention) as a 20-bit unsigned integer, and call this N_SRP.
This is the Sun Reference Pulse count.

To convert this count to time, use the formula

 T_SRP = T_RP - T_RCD + N_SRP/F_HFC

 N_SRP: Sun Reference Pulse count, as obtained above
 F_HFC: High Frequency Clock Frequency = 131072 Hz
 T_RCD: Time since datation reset pulse and TM transmission reset pulse,
 to take into account the delay between collection and sending in
 TM. This is 2 x 5.152221526 = 10.304443052 seconds, according to
 information from ESOC, and TBC by Dornier.
 (Simon Walker has pointed out that the value of the reset time
 is 5.152221526 (and not 5.150001526 as stated in the response to
 AR 28 on DDID 2.3). This is still not confirmed by Dornier.)
 T_RP: Time of Reset Pulse, obtained from the DDS packet header of the
 current housekeeping packet, according to the information on
 pages 29-30 in DDID 2.3.
 T_SRP: The resulting time of the Sun Reference Pulse.

How to obtain the instantaneous phase of the spacecraft is described in
Appendix I of DDID 2.3 (pages 108-122). I have not yet studied this
information in detail, but I believe the following is to be done.

At the time T_SRP, the spacecraft phase is Phi_SRP. At the time T, the
phase Phi is

 Phi = Phi_SRP + Omega x (T - T_SRP)

Omega is the spin rate, which may be obtained from the difference
between two successive values of T_SRP (as described in App I.6.1 of
DDID 2.3). For a higher accuracy, it may be obtained from a group of
values of T_SRP (as described in App I.6.2 of DDID 2.3). Nominally, it
should be 360/4 degrees/s.

Phi_SRP is available in the Spacecraft Attitude and Spin Rates (SATT)
file on the CD-ROM, as described on pages 76-78 of DDID 2.3. The value
in the test CD-ROM is 333.800 degrees, for all four spacecraft (look in
the files 940901ga.1a1, .1a2, .1a3, .1a4).
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If a high accuracy is required (better than 0.001 degree) corrections
should be applied as described in Appendix I of DDID 2.3. I have not
bothered to understand the details of this.

In summary, I think the only outstanding question is the "TBC by
Dornier" above. I am waiting for response from ESOC on this. If anybody
else has the answer, I would appreciate if you could let me know.

Also, I wish to underline that the above is my current understanding of
the situation. If anybody else has penetrated deeply into Appendix I of
DDID 2.3, or has some other information to complement or contradict the
above, I would be happy to hear from you.

Best regards,
Per-Arne
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APPENDIX 3

The STAFF Search Coil Axes

The data below is presented successively for the four flight models. For each model, the table shows the angles (in degrees) measured between the search coil axes and the WEC coordinate axes, at a frequency of 475 Hz (value for Cluster 1, TBC for Cluster 2). The matrix equation gives the corresponding direction cosines

$$s^i_j = \mathbf{w}^j \cdot \mathbf{s}^i \ . \qquad (61)$$

In these equations; the diagonal elements are the values required for each of the three vectors $\mathbf{s}^i$ to be of unit length. Using these values for the diagonal elements, the offset of the electrical axis with respect to the mechanical axis can be determined for each axis.

Finally the "orthogonality' of the sensor triad can be quantified by either the determinant det(**s**) (which enters into eqs. 56 *et seq.*), or by the angle θ between $\mathbf{s}^1$ and $\mathbf{s}^2 \times \mathbf{s}^3$.

Flight Model 1 *This is still Cluster 1 data*

Measurements performed at Chambon-la-Fôret on ????? (see doc ref. ?????).

	x-sensor	y-sensor	z-sensor
x-coordinate	–	90.110	90.124
y-coordinate	89.899	–	90.110
z-coordinate	89.862	89.895	–

The direction cosines (with respect to the WEC coordinate system) of the sensor axes $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are:

$$s(1)^i_j = \begin{pmatrix} s^1_1 & s^2_1 & s^3_1 \\ s^1_2 & s^2_2 & s^3_2 \\ s^1_3 & s^2_3 & s^3_3 \end{pmatrix} = \begin{pmatrix} .999996 & -.0019 & -.0022 \\ +.0018 & .999997 & -.0019 \\ +.0024 & +.0018 & .999996 \end{pmatrix} .$$

The diagonal elements are the cosines of the offsets of the sensors with respect to their nominal axes ; these are so small that they are best calculated from the off-diagonal elements):

$$B_x : 0.172°, \quad B_y : 0.159°, \quad B_x : 0.167°.$$

The orthogonality of the sensor triad is measured by

$$\det(\mathbf{s}) = .99539 \quad \text{which corresponds to} \quad \theta = 5.50° \ .$$

Flight Model 2

Measurements performed at Chambon-la-Fôret on ????? (see doc ref. ?????).

	x-sensor	y-sensor	z-sensor
x-coordinate	–	90.233	90.108
y-coordinate	89.881	–	90.202
z-coordinate	90.033	89.900	–

The direction cosines (with respect to the WEC coordinate system) of the sensor axes $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are:

$$s(2)^i_j = \begin{pmatrix} s^1_1 & s^2_1 & s^3_1 \\ s^1_2 & s^2_2 & s^3_2 \\ s^1_3 & s^2_3 & s^3_3 \end{pmatrix} = \begin{pmatrix} .999998 & -.0041 & -.0019 \\ +.0021 & .999990 & -.0035 \\ -.0006 & .0017 & .999992 \end{pmatrix} .$$

The diagonal elements yield the offsets for the sensors with respect to their nominal axes:

$$B_x : O.11°, \quad B_y : 0.25°, \quad B_x : 0.23°.$$

The orthogonality of the sensor triad is measured by

$$\det(\mathbf{s}) = .99539 \quad \text{which corresponds to} \quad \theta = 5.50° \ .$$

Flight Model 3 *This is still Cluster 1 data*

Measurements performed at Chambon-la-Fôret on ????? (see doc ref. ?????).

	x-sensor	y-sensor	z-sensor
x-coordinate	–	91.28	92.11
y-coordinate	91.08	–	92.23
z-coordinate	91.35	91.31	–

The direction cosines (with respect to the WEC coordinate system) of the sensor axes $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are:

$$s(3)^i_j = \begin{pmatrix} s^1_1 & s^2_1 & s^3_1 \\ s^1_2 & s^2_2 & s^3_2 \\ s^1_3 & s^2_3 & s^3_3 \end{pmatrix} = \begin{pmatrix} .99954 & -.0223 & -.0368 \\ -.0188 & .99949 & -.0389 \\ -.0236 & -.0229 & .99857 \end{pmatrix} .$$

The diagonal elements yield the offsets for the sensors with respect to their nominal axes:

$$B_x : 1.73°, \quad B_y : 1.83°, \quad B_x : 3.07°.$$

The orthogonality of the sensor triad is measured by

$$\det(\mathbf{s}) = .99539 \quad \text{which corresponds to} \quad \theta = 5.50° \ .$$

Flight Model 4
This is still Cluster 1 data

Measurements performed at Chambon-la-Fôret on ????? (see doc ref. ?????).

	x-sensor	y-sensor	z-sensor
x-coordinate	–	91.28	92.11
y-coordinate	91.08	–	92.23
z-coordinate	91.35	91.31	–

The direction cosines (with respect to the WEC coordinate system) of the sensor axes $\mathbf{s}^1$, $\mathbf{s}^2$ and $\mathbf{s}^3$ are:

$$s(4)^i_j = \begin{pmatrix} s^1_1 & s^2_1 & s^3_1 \\ s^1_2 & s^2_2 & s^3_2 \\ s^1_3 & s^2_3 & s^3_3 \end{pmatrix} = \begin{pmatrix} .99954 & -.0223 & -.0368 \\ -.0188 & .99949 & -.0389 \\ -.0236 & -.0229 & .99857 \end{pmatrix} .$$

The diagonal elements yield the offsets for the sensors with respect to their nominal axes:

$$B_x : 1.73°, \quad B_y : 1.83°, \quad B_x : 3.07°.$$

The orthogonality of the sensor triad is measured by

$$\det(\mathbf{s}) = .99539 \quad \text{which corresponds to} \quad \theta = 5.50° \ .$$

APPENDIX 4

Alternative derivation of Eq. 50

Alternatively, the scalar products occurring in eq. 49 may be evaluated in GEI coordinates. In this system the direction cosines **x** and **p** (by definition, and using $\alpha_p = -\pi/2$) and **h** (from eqs. 5 and 17) are

$$\mathbf{x} = \begin{pmatrix} \cos\delta_x \cos\alpha_x \\ \cos\delta_x \sin\alpha_x \\ \sin\delta_x \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} \cos\delta_p \cos\alpha_p \\ \cos\delta_p \sin\alpha_p \\ \sin\delta_p \end{pmatrix} = \begin{pmatrix} 0 \\ -\cos\delta_p \\ \sin\delta_p \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} \cos\phi_h \\ \sin\phi_h \sin\delta_p \\ \sin\phi_h \cos\delta_p \end{pmatrix}$$

and it is straightforward to evaluate the scalar products occurring in eq. 49 :

$$\left.\begin{aligned}
(\mathbf{x}\cdot\mathbf{p}) &= \sin\delta_x \sin\delta_p - \cos\delta_x \sin\alpha_x \cos\delta_p \\
(\mathbf{x}\cdot\mathbf{h}) &= \cos\delta_x(\cos\alpha_x \cos\phi_h + \sin\alpha_x \sin\phi_h \sin\delta_p) + \sin\delta_x \sin\phi_h \cos\delta_p \\
[\mathbf{x},\mathbf{p},\mathbf{h}] &= \cos\delta_p(\sin\delta_x \cos\phi_h - \cos\delta_x \cos\alpha_x \sin\phi_h \cos^2\delta_p) \\
&\qquad - \sin\delta_p \cos\delta_x(\cos\alpha_x \sin\phi_h \sin\delta_p - \sin\alpha_x \cos\phi_h) \\
&= \cos\delta_p \sin\delta_x \cos\phi_h - \cos\delta_x \cos\alpha_x \sin\phi_h(\cos^2\delta_p + \sin^2\delta_p) \\
&\qquad + \sin\delta_p \cos\delta_x \sin\alpha_x \cos\phi_h)
\end{aligned}\right\} \tag{62}$$

Eqs. 50 and 62 are formally identical.