Tentamen för kursen Rymdfysik (1FA255)
2022-10-18

Uppsala universitet
Institutionen för fysik och astronomi
Avdelningen för astronomi och rymdfysik
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Answers should be provided in Swedish or English.
Time: 08:00 - 13:00
Allowed tools: Mathematics Handbook (or equivalent), Physics Handbook, enclosed tables and formula sheets, calculator. A bilingual dictionary, for example English-Swedish or English-German, may also be used.
The exam has two parts:

- **Part A** must be satisfactorily solved in order to pass the course. You do not need to solve part A problems corresponding to examlets you have passed during the autumn 2022 course. Part A is graded by pass/fail.
- **Part B** must be solved if you aim for a grade higher than pass (3 in the Swedish 3-4-5 system, E in ECTS). Grades will depend on the number of points you score on this part, assuming you passed part A.

Part A

1. (a) Are the following statements true or false? You do not need to give any motivation or explanation, but can add comments if you feel the need to do so.
   i. As the radius of Jupiter's orbit around the sun is about five times the radius of the Earth orbit, Jupiter's orbital period is around five years.
   ii. The orbit of a satellite around Venus will to good approximation be an ellipse with the planet in one of its two focal points.
   iii. The dominant heat loss term in the thermal balance of a spacecraft usually is the emission of thermal (infrared) radiation.
   iv. Spacecraft in the solar wind always need heat shields to protect them from the high temperature of the plasma (around 10 eV, approximately 100 000 K).
   v. To change the apogee of a satellite in an economical way one should fire the thrusters at perigee.

   (b) What is the defining property of the geostationary orbit? Calculate the radius of the orbit with the same defining property around Mars.

2. (a) Are the following statements true or false? You do not need to give any motivation or explanation, but can add comments if you feel the need to do so.

   i. If "Ideal MHD" is satisfied in a particular situation, then field lines of the magnetic field will be "frozen-in" to the plasma
   ii. At 1 AU, the solar wind energy density is dominated by its thermal energy content nK_BT (rather than the magnetic energy density$B^2/2\mu_0$ or bulk kinetic energy density $mnv^2/2$).
   iii. In the Parker model, the solar wind is accelerated mainly by the $\boldsymbol{j} \times \boldsymbol{B}$ acting upon/within it.
   iv. The $\boldsymbol{j} \times \boldsymbol{B}$ force can be split into two terms - one due to tension in the electric field and the other due to thermal pressure in the plasma.
   v. The internally-produced magnetic fields of the magnetized planets can generally be treated as dipoles, since any more complex fields they possess decay faster with increasing distance.

   (b) A static flow solution for the solar wind velocity $v(r)$ can be obtained by solving

$$(v^2 - v_T^2)\frac{1}{r}\frac{dv}{dr} = 2\frac{v_T}{r} - \frac{GM}{r^2}.$$

   State the key assumptions that are applied in order to obtain this equation. What is represented by the term v_T? Sketch a solution appropriate to the solar wind, noting the point where the RHS$= 0$.

3. (a) Are the following statements true or false? You do not need to give any motivation or explanation, but can add comments if you feel the need to do so.

   i. The magnetic moment due to the gyration of a charged particle in a magnetic field is an adiabatic invariant if the magnetic field changes very little over a gyroperiod and gyroradius.
   ii. The pitch angle of a particle is a function of its temperature.
   iii. An electric field exactly perpendicular to the magnetic field (both fields homogeneous and static) causes an electric current to flow in a collisionless plasma.
   iv. For a particle moving in a dipolar magnetic field, its velocity vector is exactly aligned with the magnetic field vector at its mirror point.
   v. Over the Earth's equator, a (weak) electric current will be driven in the ionospheric plasma due to the force of gravity acting on the charges.

   (b) Illustrate and briefly explain how and why the form of the Earth's magnetic field on the equatorial plane causes ions and electrons to drift in opposite directions around the planet. In which sense does the current flow? Functionally, how will the drift period be related to the properties of the particles and the magnetic field?

4. (a) Are the following statements true or false? You do not need to give any motivation or explanation, but can add comments if you feel the need to do so.

   i. Above about 100 km altitude in the Earth's atmosphere, the density of any particle species in the neutral gas approximately decays with altitude h as $\exp(-h/H)$, where the scale height H is equal to the Debye length of the species.
   ii. The main reason for ionization in the Earth's ionosphere is the ionizing radiation (mainly in the EUV range).
   iii. The auroral light is emitted when atoms (sometimes also molecules and/or ions) in the upper atmosphere de-excite after having been excited by electrons in the keV range coming down along the magnetic field lines from the magnetosphere.
   iv. Magnetospheric substorms occur when magnetic energy has been piling up in the magnetotail during some hours of southward interplanetary magnetic field.
   v. The AE index is a measure of the strength of the equatorial ring current.

   (b) Figure 1 shows measurements collected by a spacecraft in the solar wind at L1. Panel (a) shows the magnetic field strength, panel (b) are the magnetic field components (in GSE coordinates), the solar wind speed is plotted in panel (c), and the ion number density is found in panel (d). The AE index is shown in panel (e) whereas the Dst is plotted in panel (f).

i. Using panels (a-d) only, provide an explanation as to why this data suggests that there should be strong space weather on Earth.
ii. Using panels (a-d) only, what type of phenomena in the solar wind do you think these measurements show? Also, provide reasoning for your answer.
iii. The geomagnetic indices plotted in panels (e-f) show large variations. For panels (e) and (f) separately, explain what type of magnetospheric phenomena could produce these measurements that were recorded at the ground. Also, provide reasoning for your answer.

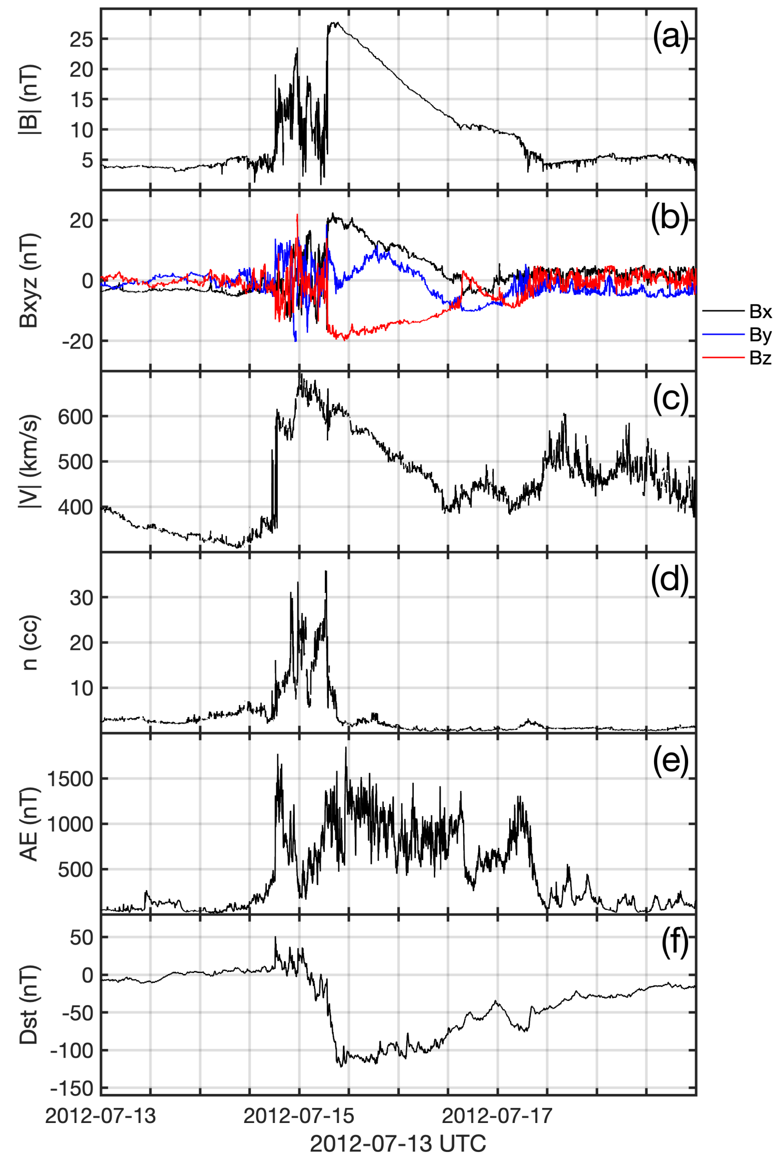

Figure 1: Solar wind measurements at L1 (a-d) and geomagnetic indices (e-f)

Part B

5. Figure 2 shows the trajectory (solid curve) of the European Rosetta spacecraft from its launch into space in March 2, 2004 to the final arrival at comet 67P/Churyumov-Gerasimenko in May 22, 2014 (which Rosetta then followed at close distance for more than two years). The reference system used in the plot is an inertial system. Dashed curves mark the orbits of Earth and Mars. Dotted curves mark the orbits of comet 67P (and some short segements of the orbits of two asteroids, Lutetia and Steins). Dates for flybys (marked with solid black circles) of planets and other objects are given. Also shown are thruster firings (open circles), with the Δv of each such maneuvre in parenthesis.

   (a) The trajectory during the first year after launch is highlighted in blue. Following the first Earth flyby (denoted "Earth 1"), the trajectory is highlighted by orange dashing. Calculate the Δv provided by the first Earth flyby. Neglect the impact of all thruster firings. (4 p)

   (b) Do you think detailed calculations including the thruster firings would have made a large change to your result above? Why? (1 p)

   (c) Why was there a need for a final orbit manoeuvre, with the thrusters providing a Δv of 794 m/s, when the spacecraft met up with the comet in May 2014? As seen in an inertial frame anchored in the Sun, was this an acceleration or deceleration of Rosetta? (1 p)

   If your solution method requires a ruler but you did not bring one, you can find one on the last page of this exam.

6. (a) Draw a sketch of the Earth's magnetosphere, viewed from the dusk side (so that the Sun is to the left of the page). Label the following regions, boundaries and features: Earth's magnetic field, magnetopause, bow shock, magnetosheath, magnetosphere, and stream lines of the solar wind. Indicate where the most significant electric currents flow through the plane of the page in this diagram. You may ignore the presence of the IMF for this diagram! (2 p)

   (b) Consider the tail lobes of the magnetosphere, some distance away from the Earth. Assume the current sheet at the center of the tail has half-thickness of 5000 km and the lobe field strength is 10 nT. Calculate the current density within the current sheet, assuming the current is uniform in strength. Why is it reasonable to expect that the magnetic force $\boldsymbol{j} \times \boldsymbol{B} = -\nabla B^2/(2\mu_0)$ in this region? (3 p)

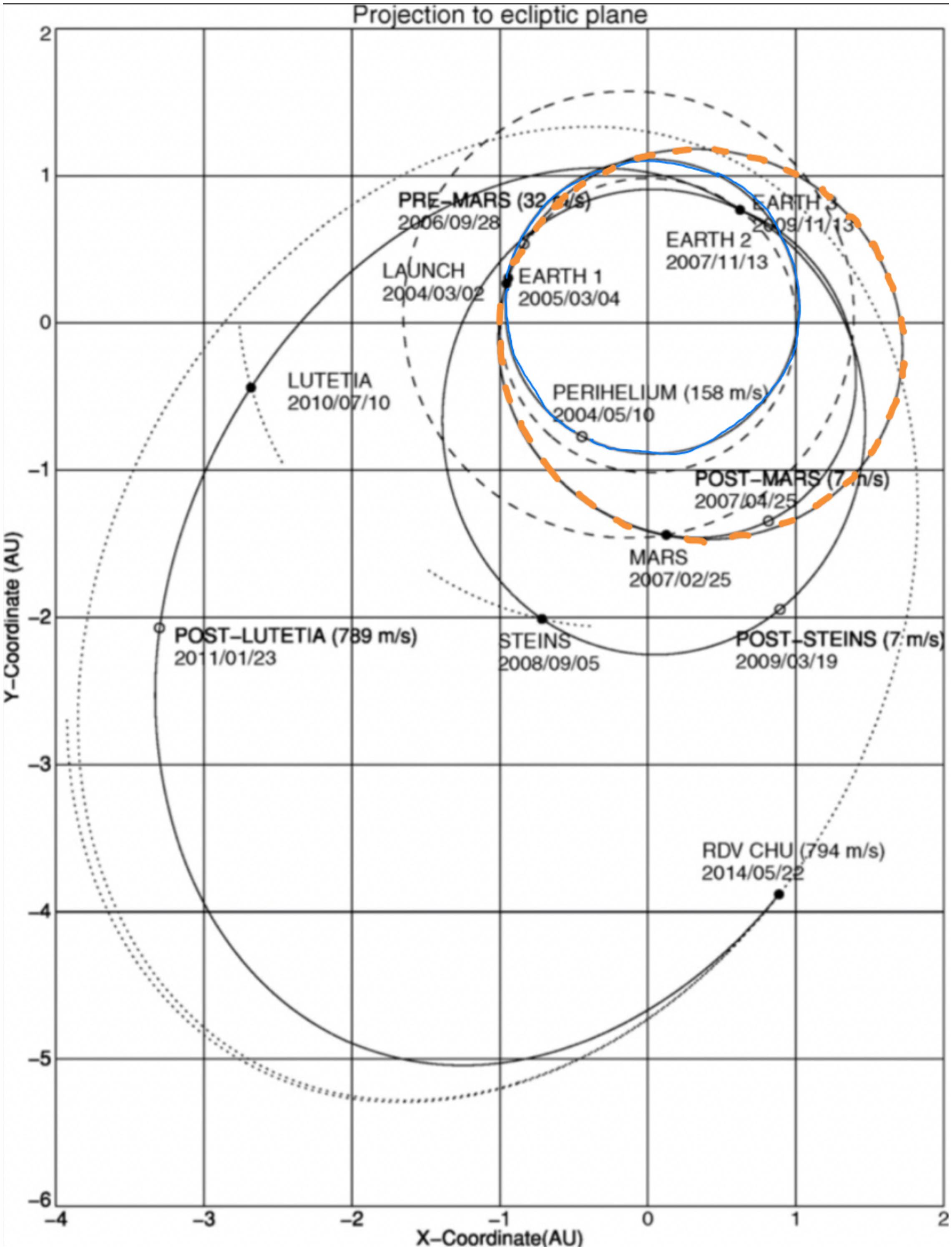

Figure 2: The Rosetta trajectory through the solar system.

7. The geomagnetic field may be taken to be a dipole field with strength 30 μT on the ground at the equator.

   (a) Explain, in words and using a diagram, how charged particles can become "trapped" on magnetic field lines, bouncing between the northern and southern hemispheres of the planet. (2 p)

   (b) Consider a particle trapped and bouncing between the two hemispheres. Show that

   $$v_{\|}(s) = v\sqrt{1 - \frac{B(s)}{B_M}},$$

   where s is a coordinate along the field line and B_M is the magnetic field at point where the particle mirrors. (1 p)

   (c) A 10 keV electron is moving on a magnetic field line with $L = 3$, and mirrors at an altitude of 2000 km. What will it's gyroradius and pitch angle be as it crosses the equator? (2 p)

8. A comet nucleus is a small (few km) solar system body containing a lot of volatile material (ices). When entering the inner solar system, the heat from the sun causes the ices to evaporate and gas (often dominated by water vapour) is emitted (creating the enormous cloud we see as a comet in the night sky). As the mass of the nucleus is small the gas is not gravitationally bound to the nucleus but escapes radially outward at almost constant speed u.

   (a) Discuss/show why the neutral gas number density should decrease with distance r from the nucleus centre as $1/r^2$ if the gas emission is isotropic, i.e. the same in every direction. To what extent do you expect the $1/r^2$ density profile to be true along any given radial direction out from the nucleus also if the gas outflow is anisotropic (for example, mostly from the sunlit side of the nucleus)? You may neglect the loss of neutrals by ionization and assume the gas is so tenuous that collisions between molecules are rare. (2 p)

   (b) The solar EUV radiation causes ionization of the comet atmosphere, leading to a comet ionopsphere. Given the neutral density profile above, derive an expression for how the solar EUV intensity I varies as a function of r along the line from the comet nucleus toward the Sun. Hint: the loss dI of EUV intensity in a thin layer between r and r+dr should be proportional to something, something and something. (2 p)

Lycka till!

Space Physics Formulas:
Complement to Physics Handbook

Charge density and current density from particle species s:

$$\rho = \sum_s q_s n_s, \qquad \mathbf{j} = \sum_s q_s n_s \mathbf{v_s}$$

Galilean transformations:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \qquad \mathbf{B}' = \mathbf{B}$$

Dipole magnetic field:

$$\mathbf{B}(r,\theta) = -B_0 \left(\frac{R_0}{r}\right)^3 \left(2\hat{\mathbf{r}}\cos\theta + \hat{\theta}\sin\theta\right)$$

Dipole field lines:

$$r/\sin^2\theta = \text{const.}$$

Magnetic field energy density and pressure:

$$w_B = p_B = \frac{B^2}{2\mu_0}$$

Equation of motion of neutral gas:

$$\rho_{\mathrm{m}} \frac{d\mathbf{v}}{dt} = -\nabla p + \text{other forces}$$

Equation of motion of gas of charged particle species s:

$$m_s n_s \frac{d\mathbf{v_s}}{dt} = n_s q_s (\mathbf{E} + \mathbf{v_s} \times \mathbf{B}) - \nabla p_s + \text{o.f.}$$

MHD equation of motion:

$$\rho_{\mathrm{m}} \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p + \text{o.f.} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \text{o.f.}$$

Equation of continuity:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = Q - L$$

Equation of state for ideal gas:

$$p = nKT$$

Dynamic pressure:

$$p_{\mathrm{dyn}} = \frac{1}{2} n m v^2$$

Condition for ”frozen-in” magnetic field:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

Ohm’s law:

$$\mathbf{j} = \begin{pmatrix} \sigma_{\mathrm{P}} & -\sigma_{\mathrm{H}} & 0 \\ \sigma_{\mathrm{H}} & \sigma_{\mathrm{P}} & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_{\parallel} \end{pmatrix} = \sigma_{\mathrm{P}} \mathbf{E}_{\perp} + \sigma_{\mathrm{H}} \frac{\mathbf{B} \times \mathbf{E}_{\perp}}{B} + \sigma_{\parallel} \mathbf{E}_{\parallel}$$

Conductivities:

$$\begin{aligned} \sigma_{\mathrm{P}} &= \tfrac{ne}{B} \left(\tfrac{\omega_{\mathrm{ci}}\nu_{\mathrm{i}}}{\omega_{\mathrm{ci}}^2 + \nu_{\mathrm{i}}^2} + \tfrac{\omega_{\mathrm{ce}}\nu_{\mathrm{e}}}{\omega_{\mathrm{ce}}^2 + \nu_{\mathrm{e}}^2} \right) \\ \sigma_{\mathrm{H}} &= \tfrac{ne}{B} \left(\tfrac{\omega_{\mathrm{ci}}^2}{\omega_{\mathrm{ci}}^2 + \nu_{\mathrm{i}}^2} - \tfrac{\omega_{\mathrm{ce}}^2}{\omega_{\mathrm{ce}}^2 + \nu_{\mathrm{e}}^2} \right) \\ \sigma_{\parallel} &= ne^2 \left(\tfrac{1}{m_{\mathrm{i}}\nu_{\mathrm{i}}} + \tfrac{1}{m_{\mathrm{e}}\nu_{\mathrm{e}}} \right) \end{aligned}$$

Cyclotron frequency (gyrofrequency):

$$f_{\mathrm{c}} = \omega_{\mathrm{c}}/(2\pi) = \frac{1}{2\pi}\frac{qB}{m}$$

Magnetic moment of charged particle gyrating in magnetic field:

$$\mu = \frac{1}{2}mv_{\perp}^2/B$$

Magnetic force on magnetic dipole:

$$\mathbf{F}_B = -\mu\nabla B$$

Drift motion due to general force $\mathbf{F}$:

$$\mathbf{v}_{\mathbf{F}} = \frac{\mathbf{F}\times\mathbf{B}}{qB^2}$$

Pitch angle:

$$\tan\alpha = v_{\perp}/v_{\parallel}$$

Electrostatic potential from charge Q in a plasma:

$$\Phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{e^{-r/\lambda_{\mathrm{D}}}}{r}$$

Debye length:

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\epsilon_0 KT}{ne^2}}$$

Plasma frequency:

$$f_{\mathrm{p}} = \omega_{\mathrm{p}}/(2\pi) = \frac{1}{2\pi}\sqrt{\frac{ne^2}{\epsilon_0 m_{\mathrm{e}}}}$$

Rocket thrust:

$$T = v_{\mathrm{e}}\frac{\mathrm{d}m}{\mathrm{d}t}$$

Specific impulse:

$$I_{\mathrm{sp}} = \frac{\int T\,\mathrm{d}t}{m_{\mathrm{fuel}}g} = v_{\mathrm{e}}/g$$

The rocket equation:

$$\Delta v = -gt_{\mathrm{burn}} + v_{\mathrm{e}}\ln\left(1 + \frac{m_{\mathrm{fuel}}}{m_{\mathrm{payload+structure}}}\right)$$

Total energy of elliptic orbit of semimajor axis a:

$$E = -\frac{GMm}{2a}$$

Kepler's third law:

$$T^2 \propto a^3$$

Emitted thermal radiation power:

$$P_{\mathrm{e}} = \varepsilon\sigma A_{\mathrm{e}}T^4$$

Absorbed solar radiation power:

$$P_{\mathrm{a}} = \alpha A_{\mathrm{a}}I_{\mathrm{rad}}$$

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Planetary data

	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO
Mass (10^{24}kg)	0.330	4.87	5.97	0.073	0.642	1898	568	86.8	102	0.0130
Diameter (km)	4879	12,104	12,756	3475	6792	142,984	120,536	51,118	49,528	2376
Density (kg/m^3)	5429	5243	5514	3340	3934	1326	687	1270	1638	1850
Gravity (m/s^2)	3.7	8.9	9.8	1.6	3.7	23.1	9.0	8.7	11.0	0.7
Escape Velocity (km/s)	4.3	10.4	11.2	2.4	5.0	59.5	35.5	21.3	23.5	1.3
Rotation Period (hours)	1407.6	-5832.5	23.9	655.7	24.6	9.9	10.7	-17.2	16.1	-153.3
Length of Day (hours)	4222.6	2802.0	24.0	708.7	24.7	9.9	10.7	17.2	16.1	153.3
Distance from Sun (10^6 km)	57.9	108.2	149.6	0.384*	228.0	778.5	1432.0	2867.0	4515.0	5906.4
Perihelion (10^6 km)	46.0	107.5	147.1	0.363*	206.7	740.6	1357.6	2732.7	4471.1	4436.8
Aphelion (10^6 km)	69.8	108.9	152.1	0.406*	249.3	816.4	1506.5	3001.4	4558.9	7375.9
Orbital Period (days)	88.0	224.7	365.2	27.3*	687.0	4331	10,747	30,589	59,800	90,560
Orbital Velocity (km/s)	47.4	35.0	29.8	1.0*	24.1	13.1	9.7	6.8	5.4	4.7
Orbital Inclination (degrees)	7.0	3.4	0.0	5.1	1.8	1.3	2.5	0.8	1.8	17.2
Orbital Eccentricity	0.206	0.007	0.017	0.055	0.094	0.049	0.052	0.047	0.010	0.244
Obliquity to Orbit (degrees)	0.034	177.4	23.4	6.7	25.2	3.1	26.7	97.8	28.3	122.5
Mean Temperature (C)	167	464	15	-20	-65	-110	-140	-195	-200	-225
Surface Pressure (bars)	0	92	1	0	0.01	Unknown*	Unknown*	Unknown*	Unknown*	0.00001
Number of Moons	0	0	1	0	2	79	82	27	14	5
Ring System?	No	No	No	No	No	Yes	Yes	Yes	Yes	No
Global Magnetic Field?	Yes	No	Yes	No	No	Yes	Yes	Yes	Yes	Unknown
	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO

Fundamental Physical Constants — Frequently used constants

Quantity	Symbol	Value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c, c_0	299 792 458	m s^{-1}	exact
magnetic constant	μ_0	$4\pi \times 10^{-7}$	N A^{-2}	
		$= 12.566\,370\,614... \times 10^{-7}$	N A^{-2}	exact
electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,817... \times 10^{-12}$	F m^{-1}	exact
Newtonian constant of gravitation	G	$6.673\,84(80) \times 10^{-11}$	m^3 kg^{-1} s^{-2}	1.2×10^{-4}
Planck constant	h	$6.626\,069\,57(29) \times 10^{-34}$	J s	4.4×10^{-8}
$h/2\pi$	$\hbar$	$1.054\,571\,726(47) \times 10^{-34}$	J s	4.4×10^{-8}
elementary charge	e	$1.602\,176\,565(35) \times 10^{-19}$	C	2.2×10^{-8}
magnetic flux quantum $h/2e$	Φ_0	$2.067\,833\,758(46) \times 10^{-15}$	Wb	2.2×10^{-8}
conductance quantum $2e^2/h$	G_0	$7.748\,091\,7346(25) \times 10^{-5}$	S	3.2×10^{-10}
electron mass	m_e	$9.109\,382\,91(40) \times 10^{-31}$	kg	4.4×10^{-8}
proton mass	m_p	$1.672\,621\,777(74) \times 10^{-27}$	kg	4.4×10^{-8}
proton-electron mass ratio	$m_\mathrm{p}/m_\mathrm{e}$	$1836.152\,672\,45(75)$		4.1×10^{-10}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5698(24) \times 10^{-3}$		3.2×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,074(44)$		3.2×10^{-10}
Rydberg constant $\alpha^2 m_\mathrm{e} c/2h$	R_∞	$10\,973\,731.568\,539(55)$	m^{-1}	5.0×10^{-12}
Avogadro constant	N_A, L	$6.022\,141\,29(27) \times 10^{23}$	mol^{-1}	4.4×10^{-8}
Faraday constant $N_\mathrm{A} e$	F	$96\,485.3365(21)$	C mol^{-1}	2.2×10^{-8}
molar gas constant	R	$8.314\,4621(75)$	J mol^{-1} K^{-1}	9.1×10^{-7}
Boltzmann constant R/N_A	k	$1.380\,6488(13) \times 10^{-23}$	J K^{-1}	9.1×10^{-7}
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3 c^2$	σ	$5.670\,373(21) \times 10^{-8}$	W m^{-2} K^{-4}	3.6×10^{-6}
		Non-SI units accepted for use with the SI		
electron volt (e/C) J	eV	$1.602\,176\,565(35) \times 10^{-19}$	J	2.2×10^{-8}
(unified) atomic mass unit $\frac{1}{12}m(^{12}\mathrm{C})$	u	$1.660\,538\,921(73) \times 10^{-27}$	kg	4.4×10^{-8}

Vector Analysis Formulae

Identities

1 • $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$

2 • $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

3 $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

4 $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=\mathbf{C}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})\}-\mathbf{D}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}$

5 $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})=\mathbf{B}\{\mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})\}-\mathbf{A}\{\mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})\}$

6 • $\nabla(fg)=f\nabla g+g\nabla f$

7 $\nabla(f/g)=(1/g)\nabla f-\left(f/g^2\right)\nabla g$

8 $\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{B}\times(\nabla\times\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{B})$

9 $\nabla\cdot(f\mathbf{A})=(\nabla f)\cdot\mathbf{A}+f(\nabla\cdot\mathbf{A})$

10 • $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

11 • $(\nabla\cdot\nabla)f=\nabla^2 f$

12 • $\nabla\times(\nabla f)=0$

13 • $\nabla\cdot(\nabla\times\mathbf{A})=0$

14 $\nabla\times(f\mathbf{A})=(\nabla f)\times\mathbf{A}+f(\nabla\times\mathbf{A})$

15 $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}+(\nabla\cdot\mathbf{B})\mathbf{A}-(\nabla\cdot\mathbf{A})\mathbf{B}$

16a • $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

16b $\nabla^2\mathbf{A}=\nabla\cdot(\nabla\cdot\mathbf{A})-\nabla\times(\nabla\times\mathbf{A})$

17 • $\nabla(1/r)=-\hat{\mathbf{r}}/r^2$

If S is the closed surface that encloses the volume V and C is the closed curve that bounds an open surface A then:

18 • $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot \mathrm{d}\mathbf{l}=f(\mathbf{b})-f(\mathbf{a})$

19 $\int_V(\nabla f)\,\mathrm{d}V=\oint_S f\;\mathrm{d}\mathbf{S}$

20 $\int_V(\nabla\times\mathbf{B})\mathrm{d}V=-\oint_S\mathbf{B}\times\mathrm{dS}$

21 • $\oint_S\mathbf{B}\cdot\mathrm{d}\mathbf{S}=\int_V(\nabla\cdot\mathbf{B})\,\mathrm{d}V$ (The Divergence Theorem)

22 • $\oint_C\mathbf{B}\cdot\mathrm{d}\mathbf{l}=\int_A(\nabla\times\mathbf{B})\cdot\mathrm{d}\mathbf{A}$ (Stokes's Theorem)

Special Coordinate Systems

Cartesian Coordinates (x, y, z)

- $\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$
- $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- $\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$
- $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
- $\nabla^2 \mathbf{A} = \nabla^2 A_x \hat{\mathbf{x}} + \nabla^2 A_y \hat{\mathbf{y}} + \nabla^2 A_z \hat{\mathbf{z}} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$

Cylindrical Polar Coordinates (r, θ, z)

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Polar Coordinates (r, θ, φ)

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\varphi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

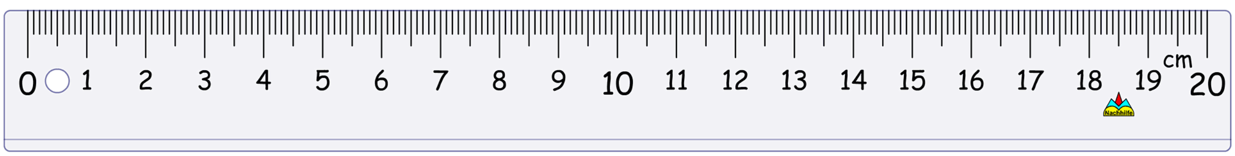

Figure 3: If you did not bring your own ruler, detach this paper and fold it along the upper edge of the image to get a nice working ruler.