T12I022/1)

1:1	A B C
1:2	A B C
1:3	– – C
1:4	– B –
1:5	A – –
1:6	– – –
1:7	– – C
1:8	A B C
1:9	A B –
1:10	A B C

T 12 10 22/2)

(a) From

$$\bar{B}(\bar{r}) = \begin{cases} -B_0 \hat{x}, & z < -a \\ B_0 \frac{3a^2 z - z^3}{2a^3} \hat{x}, & |z| \leq a \\ B_0 \hat{x}, & z > a \end{cases} \quad (1)$$

we can calculate the current density using Ampère's law,

$$\nabla \times \bar{B} = \mu_0 \bar{J} \quad (2)$$

$$\Rightarrow \quad \bar{J} = \begin{cases} 0, & |z| > a \\ \frac{1}{\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B(z) & 0 & 0 \end{vmatrix}, & |z| \leq a \end{cases} \quad (3)$$

Concentrating on the region $|z| \leq a$, we have

$$\bar{J} = \frac{1}{\mu_0} \hat{y} \frac{\partial B(z)}{\partial z} = \frac{\hat{y} B_0}{\mu_0} \frac{\partial}{\partial z}\left(\frac{3a^2 z - z^3}{2a^3}\right) =$$

$$= \frac{\hat{y} B_0}{\mu_0} \frac{3a^2 - 3z^2}{2a^3} = \frac{3B_0}{2\mu_0 a} \frac{a^2 - z^2}{a^2} \hat{y} \quad (4)$$

As $\bar{J} = 0$ outside $|z| < a$, the force density must be zero there. Inside $|z| < a$, we get from (1) & (4)

$$\bar{J} \times \bar{B} = \frac{3B_0^2}{2\mu_0 a} \cdot \frac{a^2 - z^2}{a^2} \cdot \frac{3a^2 z - z^3}{2a^3} \hat{y} \times \hat{x} =$$

$$= -\frac{3B_0^2}{4\mu_0 a^6} (a^2 - z^2)(3a^2 - z^2) z \hat{z} \quad (5)$$

At $z = 0$, we have from (4) & (5) that

$$\bar{j} = \frac{3B_0}{2\mu_0 a}\hat{y} = \frac{3\cdot 1\cdot 10^{-9}}{2\cdot 4\pi\cdot 10^{-7}\cdot 2\cdot 10^{6}}\hat{y}\ A/m^2$$

$$\approx 0.6\hat{y}\ nA/m^2 \tag{6}$$

and

$$\bar{j}\times\bar{B} = 0 \tag{7}$$

To summarize:

$$\bar{j}(\bar{r}) = \begin{cases} 0, & |z| > a \\ \dfrac{3B_0}{2\mu_0 a}\dfrac{a^2-z^2}{a^2}\hat{y}, & |z| \leq a \end{cases}$$

$$\bar{j}\times\bar{B} = \begin{cases} 0, & |z| > a \\ -\dfrac{3B_0^2}{4\mu_0 a^6}(a^2-z^2)(3a^2-z^2)z\,\hat{z}, & |z| \leq a \end{cases}$$

$$\bar{j}(z=0) \approx 0.6\,\hat{y}\ nA/m^2$$

$$\bar{j}\times\bar{B}(z=0) = 0$$

(b) The magnetic energy density is

$$w_B = \frac{B^2}{2\mu_0}, \tag{8}$$

so the total magnetic energy in the volume is

$$W_B = \int_{-a}^{a}\int_{-10a}^{10a}\int_{-a}^{a}\frac{B^2}{2\mu_0}\,dx\,dy\,dz. \tag{9}$$

From (1), $\bar{B}$ only depends on z, so the integrations over x and y are trivial. We get

$$W_B = 2a \cdot 20a \int_{-a}^{a} \frac{B^2(z)}{2\mu_0}\, dz \tag{10}$$

Using (1), (10) gives

$$W_B = \frac{20a^2 B_0^2}{\mu_0} \int_{-a}^{a} \left[\frac{3a^2 z - z^3}{2a^3}\right]^2 dz =$$

$$= \frac{5B_0^2}{\mu_0 a^4} \int_{-a}^{a} \left(9a^4 z^2 - 6a^2 z^4 + z^6\right) dz =$$

$$= \frac{5B_0^2}{\mu_0 a^4} \left[3a^4 z^3 - \frac{6}{5} a^2 z^5 + \frac{1}{7} z^7\right]_{-a}^{a} =$$

$$= \frac{5B_0^2}{\mu_0 a^4} \left[6a^7 - \frac{12}{5} a^7 + \frac{2}{7} a^7\right] =$$

$$= \frac{10 B_0^2 a^3}{\mu_0} \left(3 - \frac{6}{5} + \frac{1}{7}\right) = \frac{680\, B_0^2 a^3}{35\,\mu_0} =$$

$$= \frac{680 \cdot (1 \cdot 10^{-9})^2 \cdot (2 \cdot 10^6)^3}{35 \cdot 4\pi \cdot 10^{-7}}\ \mathrm{J} \approx 0.12\ \mathrm{GJ} \tag{11}$$

If the cross-tail current cannot flow in this region, it will close through the ionosphere instead, via field-aligned currents connecting the tail to the ionosphere. A release of this stored magnetic energy during ~100 s (a few minutes) gives a power output ~1 MW. The scenario is relevant for a magnetospheric substorm, except for the low value of B_0 given: with $B_0 \sim 30$ nT instead, we would have had ~1 GW, more typical of a substorm.

T121022/3)

(a) The kinetic energy of a particle is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\bar{v}\cdot\bar{v}, \tag{1}$$

so its rate of change is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\bar{v}\cdot\bar{v}\right) = m\bar{v}\cdot\frac{d\bar{v}}{dt}. \tag{2}$$

But the force on a charge in a $\bar{B}$-field is

$$\bar{F} = q\bar{v}\times\bar{B}, \tag{3}$$

so we get

$$m\frac{d\bar{v}}{dt} = \bar{F} = q\bar{v}\times\bar{B}, \tag{4}$$

which in (2) gives

$$\frac{dK}{dt} = \bar{v}\cdot q(\bar{v}\times\bar{B}) = q\,\bar{v}\cdot(\bar{v}\times\bar{B}) = 0. \tag{5}$$

Thus K is constant, Q.E.D.

(b) A charged particle in a dipole field has three characteristic periodicities of motion, if its gyroradius is much smaller than the characteristic dimensions of the field:

(i) The gyroperiod, for gyration around $\bar{B}$

(ii) The bounce period, for motion along $\bar{B}$ between mirror points

(iii) The drift period, for drift around the field due to ∇B.

A particle with 90° pitch angle in the equatorial plane sees a zero magnetic field gradient along $\bar{B}$, and will therefore always stay in this plane. There is no bounce motion in this case, so only (i) and (iii) are relevant.

For symmetry reasons, the drift orbit will be a circle in the equatorial plane, so the particle will stay at the same geocentric distance r all the time, so it encounters a constant B and will have the same gyroperiod all the time. From the expression for a dipole field,

$$\bar{B}(r,\theta) = -B_0\left(\frac{R_E}{r}\right)^3(2\hat{r}\cos\theta + \hat{\theta}\sin\theta), \tag{6}$$

we get the B value in the eq. plane at 3 R_E as

$$B = B_0\left(\frac{1}{3}\right)^3 = B_0/27. \tag{7}$$

We thus get the gyroperiod as

$$T_g = \frac{2\pi}{\omega_c} = \frac{2\pi m}{qB} = \frac{54\pi m}{eB_0} =$$

$$= \frac{54\cdot\pi\cdot 16\cdot 1{,}67\cdot 10^{-27}}{1{,}6\cdot 10^{-19}\cdot 30\cdot 10^{-6}}\ s \approx 0.94\ s. \tag{8}$$

The drift period can be found from the length of a full path around the Earth,

$$L = 2\pi r = 6\pi R_E, \tag{9}$$

and the drift speed

$$v_{\nabla B} = |\bar{v}_{\nabla B}| \tag{10}$$

as the drift velocity $\bar{v}_{\nabla B}$ should be azimuthal. From the general force drift,

$$\bar{v}_F = \frac{\bar{F}\times\bar{B}}{qB^2} \tag{11}$$

the case of a gradient force on a magnetic dipole

$$\bar{F} = -\mu\nabla B \tag{12}$$

gives

$$\bar{v}_{\nabla B} = \frac{\mu}{q}\,\frac{\bar{B}\times\nabla B}{B^2}. \tag{13}$$

From (6),

$$B = B_0 \left(\frac{R_E}{r}\right)^3 \sqrt{4\cos^2\theta + \sin^2\theta} \qquad (14)$$

and (in the eq. plane, where $\theta = 90°$)

$$\bar{B} = -B_0 \left(\frac{R_E}{r}\right)^3 \hat{\theta} = -B\hat{\theta}. \qquad (15)$$

As $\bar{B}$ is along $\hat{\theta}$, only the $\hat{r}$ component of ∇B will be of interest in (13). By (14), this is

$$\hat{r}\frac{\partial B}{\partial r} = -3B_0 \frac{R_E^3}{r^4}\hat{r} = -\frac{3}{r}B\hat{r} \qquad (16)$$

for $\theta = 90°$, which in (13) gives

$$\bar{v}_{\nabla B} = \frac{\mu}{qB^2}\bar{B} \times \hat{r}\frac{\partial B}{\partial r} = \frac{\mu}{qB^2}(-B\hat{\theta}) \times \left(-\frac{3}{r}B\hat{r}\right) =$$

$$= -\frac{3\mu}{qr}\hat{\varphi} \qquad (17)$$

where we also used (15) and $\hat{\theta} \times \hat{r} = -\hat{\varphi}$. The expectation that the drift is azimuthal is thus confirmed.

We get

$$v_{\nabla B} = |\bar{v}_{\nabla B}| = \frac{3\mu}{qr}, \qquad (18)$$

which with (9) gives us the drift period as

$$T_D = \frac{L}{v_{\nabla B}} = \frac{6\pi R_E q r}{3\mu} = \frac{6\pi q R_E^2}{\mu}. \qquad (19)$$

Now the magnetic moment due to gyromotion is

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B}. \qquad (20)$$

As the pitch angle is 90°, all kinetic energy is in the gyromotion, and $\frac{1}{2}mv_\perp^2 = 10$ keV. With (7), (19) gives

$$T_D = \frac{6\pi q R_E^2 B}{\frac{1}{2}mv_\perp^2} = \frac{6\pi q R_E^2 B_0}{27 \cdot \frac{1}{2}mv_\perp^2} = \frac{2\pi \cdot 1{,}6 \cdot 10^{-19} \cdot (6371{,}2 \cdot 10^3)^2 \cdot 30 \cdot 10^{-6}}{9 \cdot 10 \cdot 10^3 \cdot 1{,}6 \cdot 10^{-19}} \text{ s}$$

$$\approx 8{,}5 \cdot 10^4 \text{ s} \approx 23{,}6 \text{ h}. \qquad (21)$$

To be sure that the method is appropriate, we may wish to compare the scale length of B to the gyroradius. The first can be taken to be

$$L_B \sim \frac{B}{|\nabla B|} = \left|\frac{B}{\partial B/\partial r}\right| = \frac{r}{3} = R_E \qquad (22)$$

from (16) with use of $r = 3R_E$. The gyroradius is

$$r_g = \frac{v_\perp T_g}{2\pi} = \frac{\sqrt{\frac{1}{2}mv_\perp^2 \cdot \frac{2}{m}}}{2\pi} T_g =$$

$$\approx \frac{0.94}{2\pi}\sqrt{10\cdot 10^3 \cdot 1{,}6\cdot 10^{-19} \cdot \frac{2}{16\cdot 1{,}67\cdot 10^{-27}}} \text{ m} \approx \qquad (24)$$

$$\approx 37 \text{ km}$$

which is much less than $L_B \sim 6371{,}2$ km. The basic assumption $r_g \ll L_B$ is thus verified.

Answer: The two characteristic periods are the gyroperiod (0.94 s) and the drift period (23,6 h).

T121022/4)

(a) In the solar wind, the dominant pressure term is the dynamic pressure, as the solar wind is supersonic. In the magnetosphere, the magnetic pressure dominates. We therefore take the distance to the subsolar point on the magnetosphere to be given by a pressure balance

$$p_{dyn}^{SW} \approx p_B^M . \qquad (1)$$

Here

$$p_B^M = \frac{B^2}{2\mu_0} \approx \frac{B_0^2}{2\mu_0}\left(\frac{R_E}{r}\right)^6 \qquad (2)$$

if assuming a dipole field and that the subsolar point is not too far from the equatorial plane, and

$$p_{dyn}^{SW} \approx m n v_{sw}^2 . \qquad (3)$$

Taking $m \approx m_p$ for the solar wind particle mass, we approximate the lowest and highest solar wind pressures to take place around the end of the plot ($n \approx 1\ cm^{-3}$, $v_{sw} \approx 400\ km/s$) and at the point of highest n ($n \approx 80\ cm^{-3}$, $v_{sw} \approx 600\ km/s$). From (2) & (1), the distance we look for is

$$r \sim R_E \left[\frac{B_0^2}{2\mu_0 m n v_{sw}^2}\right]^{1/6}$$

$$r_{min} \sim R_E \left[\frac{(30\cdot 10^{-6})^2}{2\cdot 4\pi\cdot 10^{-7}\cdot 1{,}67\cdot 10^{-27}\cdot 80\cdot 10^6\cdot (600\cdot 10^3)^2}\right]^{1/6} \approx \underline{\underline{4.4\ R_E}}$$

$$r_{max} \sim r_{min}\cdot\left(\frac{80}{1}\right)^{1/6}\cdot\left(\frac{600}{400}\right)^{1/3} \approx \underline{\underline{10.5\ R_E}}$$

(b) The magnetopause linear dimension should scale with the distance we found in (a), i.e.

$$r \propto \frac{1}{n_{sw}^{1/6} \, v_{sw}^{1/3}} \, . \qquad (1)$$

The area over which the solar wind can interact with the magnetosphere should thus scale as

$$r^2 \propto n_{sw}^{-1/3} \, v_{sw}^{-2/3} \qquad (2)$$

The energy flux in the solar wind should scale as $n_{sw} v_{sw}^3$ (energy density times flow speed), so we could expect the energy input to the magnetosphere scales as

$$r^2 \cdot n_{sw} \cdot v_{sw}^3 \propto n_{sw}^{2/3} \, v_{sw}^{7/3} \qquad (3)$$

meaning the effect of magnetopause shrinking is small in comparison to the increase of available energy.

T 12/022/5)

(a) The energy in an orbit is given by

$$E = -\frac{GMm}{2a}. \qquad (1)$$

For motion in the solar system, M is the solar mass,

$$M = m_{\odot} = 2 \cdot 10^{30}\ \mathrm{kg}. \qquad (2)$$

The first orbit is close to a circle with radius 1 AU, so the energy in this orbit is

$$E_1 = -\frac{GMm}{2 \cdot 1\,\mathrm{AU}}. \qquad (3)$$

The final orbit to Jupiter is an ellipse with perihelion at Earth (1 AU) and apohelion at Jupiter (~5 AU). Its semimajor axis thus is

$$a = \frac{1+5}{2}\,\mathrm{AU} = 3\ \mathrm{AU} \qquad (4)$$

so that the final energy is

$$E_2 = -\frac{GMm}{2 \cdot 3\,\mathrm{AU}}. \qquad (5)$$

The energy gain from all flybys thus is

$$E_2 - E_1 = \frac{GMm}{2\,\mathrm{AU}}\left(1 - \frac{1}{3}\right) = \frac{GMm}{3\,\mathrm{AU}} =$$

$$= \frac{6.67 \cdot 10^{-11} \cdot 2 \cdot 10^{30} \cdot 4800}{3 \cdot 1.5 \cdot 10^{11}}\ \mathrm{J} \approx \underline{\underline{1.4\ \mathrm{TJ}}}$$

(This corresponds to a speed gain of 24 km/s from the four flybys)

(b) In equilibrium, the s/c heat balance is

$$P_a + P_i = P_e \qquad (1)$$

where the three terms represent power absorbed, internally generated, and emitted, respectively. We have

$$P_a = \alpha A_a I \qquad (2)$$

and

$$P_e = \varepsilon A_e \sigma T^4 \qquad (3)$$

so with no internal heating,

$$\frac{\alpha}{\varepsilon} = \frac{A_e}{A_a} \frac{\sigma T^4}{I}. \qquad (4)$$

In our case, $T = 30°C$ at 1 AU, where $I = 1.4$ kW/m²; and the absorbing and emissive areas are

$$A_a = \text{one cube side} = d^2$$

$$A_e = \text{all six cube sides} = 6d^2$$

where $d = 2$ m, so we get

$$\frac{\alpha}{\varepsilon} = 6 \frac{\sigma T^4}{I} = 6 \cdot \frac{5.67 \cdot 10^{-8} \cdot 303^4}{1.4 \cdot 10^3} \approx 2.05. \qquad (5)$$

Jupiter is at ~5 AU, so there we have $I \approx \frac{1.4}{25}$ kW/m² or 56 W/m². To keep the same temperature, the internal heating must therefore supply a power

$$P_i = \alpha A_a (I^{Earth} - I^{Jupiter}) \approx \frac{24}{25} \alpha A_a I^{Earth} \qquad (6)$$

No values are given for α or ε, but both must be in the span [0, 1]. As α is the bigger of the two according to (5), we get an upper limit to the heat needed by putting $\alpha = 1$. Hence,

$$P_i < \frac{24}{25} d^2 I^{Earth} = \frac{24}{25} \cdot 4 \cdot 1400 \text{ W} \approx \underline{5.4 \text{ kW}} \text{ (upper limit)}$$