T021219/2)

(a) The only force on the particle is the Lorentz force, so the equation of motion is

$$m\frac{d\bar{v}}{dt} = q\,\bar{v}\times\bar{B}$$

The time rate of change of the kinetic energy then is

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{1}{2}m\bar{v}\cdot\bar{v}\right) = m\bar{v}\cdot\frac{d\bar{v}}{dt} =$$

$$= \bar{v}\cdot q\,(\bar{v}\times\bar{B}) = 0$$

QED

(b) We can consider the particle as a magnetic dipole μ, which means we do not have to take care of the gyromotion. In this description, the only force on the particle is

$$\bar{F} = -\mu\nabla B. \qquad (3)$$

For a dipole field

$$\bar{B} = -B_0\left(\frac{R}{r}\right)^3(2\hat{r}\cos\theta + \hat{\theta}\sin\theta), \qquad (4)$$

we get

$$B = B_0\left(\frac{R}{r}\right)^3\sqrt{4\cos^2\theta + \sin^2\theta} =$$

$$= B_0\left(\frac{R}{r}\right)^3\sqrt{4 - 3\sin^2\theta} \qquad (5)$$

It is clear that this expression has a minimum in the equatorial plane ($\theta = 90°$), so here we must have $\frac{\partial B}{\partial \theta} = 0$ and hence

$$\nabla B \Big|_{\theta=90°} \parallel \hat{r}. \qquad (6)$$

On the other hand, (4) gives for $\theta = 90°$ that

$$\bar{B}(r, 90°) = -B_0 \left(\frac{R}{r}\right)^3 \hat{\theta} \doteq -B(r, 90°)\,\hat{\theta} \qquad (7)$$

so combining (6) & (7), we see there is no force along $\bar{B}$. The particle will thus stay in the equatorial plane forever.

Its motion here will be given by the drift

$$\bar{v} = \frac{1}{q}\frac{\bar{F} \times \bar{B}}{B^2} = \frac{\mu}{q}\frac{\bar{B} \times \nabla B}{B^2} \qquad (8)$$

As there will be no θ-component of $\nabla B \big|_{\theta=90°}$ according to (6), we have from (5) that

$$\nabla B \Big|_{\theta=90°} = \hat{r}\,\frac{\partial B}{\partial r}\Big|_{\theta=90°} = -3B_0 \frac{R^3}{r^4}\sqrt{4 - 3\sin^2\theta}\,\Big|_{\theta=90°}\,\hat{r} =$$

$$= -\frac{3}{r} B_0 \left(\frac{R^3}{r^4}\right)\hat{r} = -\frac{3}{r} B(r, 90°)\,\hat{r} \qquad (9)$$

(7) & (9) $\Rightarrow$

$$\bar{B} \times \nabla B = -B\hat{\theta} \times \left(-\frac{3}{r} B \hat{r}\right) = \frac{3B^2}{r}\,\hat{\theta} \times \hat{r} =$$

$$= -\frac{3B^2}{r}\,\hat{\varphi} \qquad (10)$$

(10) in (8) $\Rightarrow$

$$\bar{v} = -\frac{3\mu}{qr}\hat{\varphi} \qquad (11)$$

Now

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{K_\perp}{B} \qquad (12)$$

$$\Rightarrow \quad \bar{v} = -\frac{3K_\perp}{qrB}\hat{\varphi} = -\frac{3K_\perp}{qrB_0}\left(\frac{r}{R}\right)^3\hat{\varphi} =$$

$$= -\frac{3K_\perp}{qRB_0}\left(\frac{r}{R}\right)^2\hat{\varphi} \qquad (13)$$

where $K_\perp$ is the perpendicular kinetic energy of the particle, given as 1 keV.

The time to complete a full orbit will be

$$T = \frac{2\pi r}{v} = \frac{2\pi r q R B_0}{3K_\perp}\left(\frac{R}{r}\right)^2 =$$

$$= \frac{2\pi R^2 q B_0}{3K_\perp}\left(\frac{R}{r}\right) \qquad (14)$$

which in our case becomes $(r = 4R)$

$$T = \frac{2\pi(6371{,}2\cdot 10^3)^2\cdot \cancel{1{,}6\cdot 10^{-19}}\cdot 30\cdot 10^{-6}}{3\cdot 10^3\cdot \cancel{1{,}6\cdot 10^{-19}}\cdot 4}\ s =$$

$$= \frac{\pi}{2}\cdot 6{,}37^2\cdot 10^4\ s \approx 6.4\cdot 10^5\ s$$

$$\underline{\underline{\approx 1\ \text{week}}} \qquad (15)$$

T021219/3)

Given solar wind:

$$\bar{v}(r) = V_0\left(1+\ln\frac{r}{R}\right)\hat{r} \quad , \quad r > R \tag{1}$$

(a) Conservation of particles is expressed by the equation of continuity:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\bar{v}) = 0 \tag{2}$$

Assuming a stationar solution, $\frac{\partial n}{\partial t} = 0$, and for $\bar{v} \parallel \hat{r}$ we get

$$0 = \nabla\cdot(n\bar{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n v) \tag{3}$$

$$\Rightarrow \quad r^2 n(r) v(r) = \text{const} = R^2 n_0 V_0 \tag{4}$$

$$\Rightarrow \quad n(r) = n_0\left(\frac{R}{r}\right)^2 \frac{V_0}{V(r)} =$$

$$= n_0\left(\frac{R}{r}\right)^2 \frac{1}{1+\ln\frac{r}{R}} \tag{5}$$

The velocity was given only for $r \geq R$, so that is the region of validity of (5) as well.

$$\boxed{n(r) = n_0\left(\frac{R}{r}\right)^2 \frac{1}{1+\ln\frac{r}{R}} \quad , \quad r \geq R} \tag{6}$$

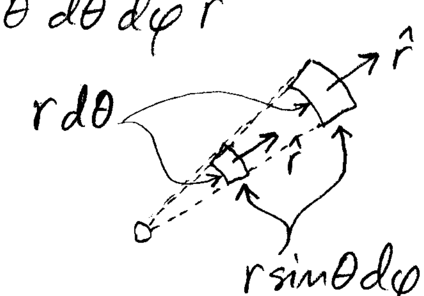
(b) As the electric field in the rest frame of the solar wind is zero ($\bar{E}+\bar{v}\times\bar{B}=0$), the magnetic field is frozen into the solar wind. The magnetic flux throug any area moving with the solar wind is therefore constant:

$$d\Phi = \bar{B}\cdot d\bar{S} = \text{const} \qquad (7)$$

To study the evolution of the three components of $\bar{B}$, we study surfaces with normal vectors along the three unit vectors. As the solar wind profile (1) is purely radial and spherically symmetric, the normal vectors of these areas will not change as the wind expands.

B_r: Area element $d\bar{S} = r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$ (8)

$$\Rightarrow \text{const} = d\Phi =$$
$$= \bar{B}\cdot d\bar{S} =$$
$$= r^2 \sin\theta \, d\theta \, d\varphi \, B_r \qquad (9)$$



In the ecliptical plane, $\sin\theta = 1$ and $B_r(R) = A$, so we get

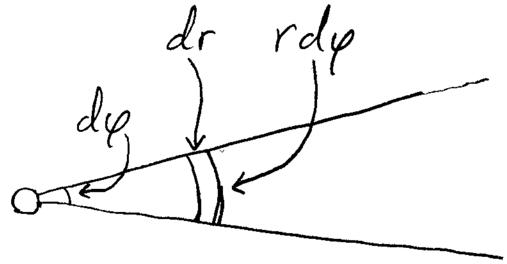
$$r^2 d\theta \, d\varphi \, B_r = R^2 d\theta \, d\varphi \, A \qquad (10)$$

$$\boxed{B_r(r, 90^\circ, \varphi) = \left(\frac{R}{r}\right)^2 A, \quad r \geq R} \qquad (11)$$

To get at B_φ, we study an area element

$$d\bar{S} = r\,dr\,d\theta\,\hat{\varphi} \tag{12}$$

The solar wind speed is not constant, so dr will depend on r. As long as the wind is stationary $\left(\frac{\partial}{\partial t}=0\right)$, the time dt it takes for a certain lump of plasma to pass by an observer at rest at arbitrary r will be constant. The radial size of the lump is clearly



$$dr = v_r(r)\,dt. \tag{13}$$

Using this in (12) and (7), we get

$$\text{const} = d\Phi = \bar{B}\cdot d\bar{S} = B_\varphi\, r\, v_r(r)\,dt\,d\theta \tag{14}$$

Putting in values for $r=R$, we get

$$B_\varphi(r)\, r\, v_r(r)\,dt\,d\theta = C R v_0\,dt\,d\theta \tag{15}$$

$$\Rightarrow \quad B_\varphi(r) = C\,\frac{R}{r}\,\frac{v_0}{v(r)} \tag{16}$$

which with (1) leads to

$$\boxed{B_\varphi(r, 90°, \varphi) = C\,\frac{R}{r}\,\frac{1}{1+\ln\frac{r}{R}}}\ , \quad r \geq R \tag{17}$$

(c) Characteristics of the given velocity profile $\bar{v}(\bar{r}) = V_0 (1 + \ln \frac{r}{R}) \hat{r}$ are as follows:

I) Direction: purely radial. This compares well with the real solar wind.

II) Symmetry: The given profile is spherically symmetric. The real solar wind is faster over the polar areas than in the ecliptical plane.

III) Dependence on r: The given profile corresponds to a constantly accelerated solar wind, though the acceleration decreases with distance:

$$\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} = 0 + v \frac{\partial v}{\partial r} =$$

$$= V_0 \left(1 + \ln \frac{r}{R}\right) \cdot \frac{1}{r} = \frac{V_0}{r}\left(1 + \ln \frac{r}{R}\right)$$

This is in fact qualitatively the case also with the real solar wind, and it may be that by tuning the parameter R one may get a reasonable model of the actual solar wind, at least over some range of r.

T 021219/4)

(a) Equation of motion of a neutral gas in a constant gravitational field:

$$mn\frac{d\bar{v}}{dt} = -\nabla p + mn\bar{g} \qquad (1)$$

Assume static equilibrium: $\bar{v} = 0$, $\frac{d}{dt} = 0$

Assume ideal gas: $p = nKT$

Assume constant temperature: $\nabla p = KT\nabla n$

Assume horizontal stratification: $\bar{g} = -g\hat{z}$

$$\nabla = \hat{z}\frac{d}{dz}$$

(1) then becomes

$$0 = -\hat{z}KT\frac{dn}{dz} - mng\hat{z} \qquad (2)$$

$$\frac{dn}{n} = -\frac{mg}{KT}dz \qquad (3)$$

$$\ln n = c - \frac{mg}{KT}z \qquad (4)$$

$$n = e^c \exp\left(-\frac{mgz}{KT}\right) \qquad (5)$$

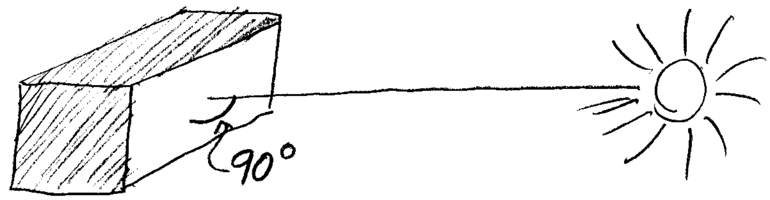
Denoting the density on the ground $(z=0)$ by n_0,

$$n = n_0 \exp\left(-\frac{mgz}{KT}\right) \qquad (6)$$

This shows the desired features:

- Exponential density decrease with altitude
- That heavier species decreases faster with altitude.

T021219/5)



(a) In equilibrium with no internal heating, we should have

$$P_a = P_e \qquad (1)$$

Where the absorbed and emitted powers are

$$P_a = \alpha A_a I \qquad (2)$$

$$P_e = \varepsilon A_e \sigma T^4 \qquad (3)$$

All values are independent on position except T and the solar intensity I, for which

$$I \propto r^{-2} \qquad (4)$$

where r is the distance from the sun. From (1),

$$\frac{P_a(5\,AU)}{P_a(1\,AU)} = \frac{P_e(5\,AU)}{P_e(1\,AU)} \qquad (5)$$

which with (2)-(4) becomes

$$\frac{1}{25} = \left(\frac{T(5\,AU)}{T(1\,AU)}\right)^4 \qquad (6)$$

With $T(1\,AU) = 40°C$ we get

$$T(5\,AU) = \frac{1}{\sqrt{5}}\,T(1\,AU) = \frac{273+40}{\sqrt{5}}\,K \qquad (7)$$

$$\boxed{T = 140\,K = -133\,°C}$$

(b) When there is internal heating, the power balance is

$$P_e = P_a + P_{in} \qquad (8)$$

The internal power needed to raise the temperature from $-133°C$ (when $P_{in} = 0$) to $0°C$ then is

$$P_{in} = P_e(0°C) - P_e(-133°C) \qquad (9)$$

as P_a is independent of temperature. We have

$$A_e = \text{total area} = 2\cdot(2\times 2)\ m^2 + 4\cdot(2\times 3)\ m^2 =$$
$$= 32\ m^2,$$

so we get from (3) and (9) that

$$P_{in} = 0.3\cdot 32\cdot 5{,}67\cdot 10^{-8}\left[273^4 - 140^4\right]\ W$$

$$\boxed{P_{in} = 2.8\ kW}$$

(c) Comet nucleus mass:

$$M = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi\cdot(500)^3\cdot 500\ kg \approx 2.6\cdot 10^{11}\ kg \qquad (1)$$

In circular motion, we must have

$$\frac{mv^2}{r} = \frac{GmM}{r^2} \qquad (2)$$

$$\Rightarrow v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67\cdot 10^{-11}\cdot 2.6\cdot 10^{11}}{1000+500}}\ m/s \approx \underline{\underline{11\ cm/s}}$$

Period: $T = \frac{2\pi r}{v} = \frac{2\pi\cdot(1000+500)}{0.11}\ s \approx 87600\ s \approx \underline{1\ day}$