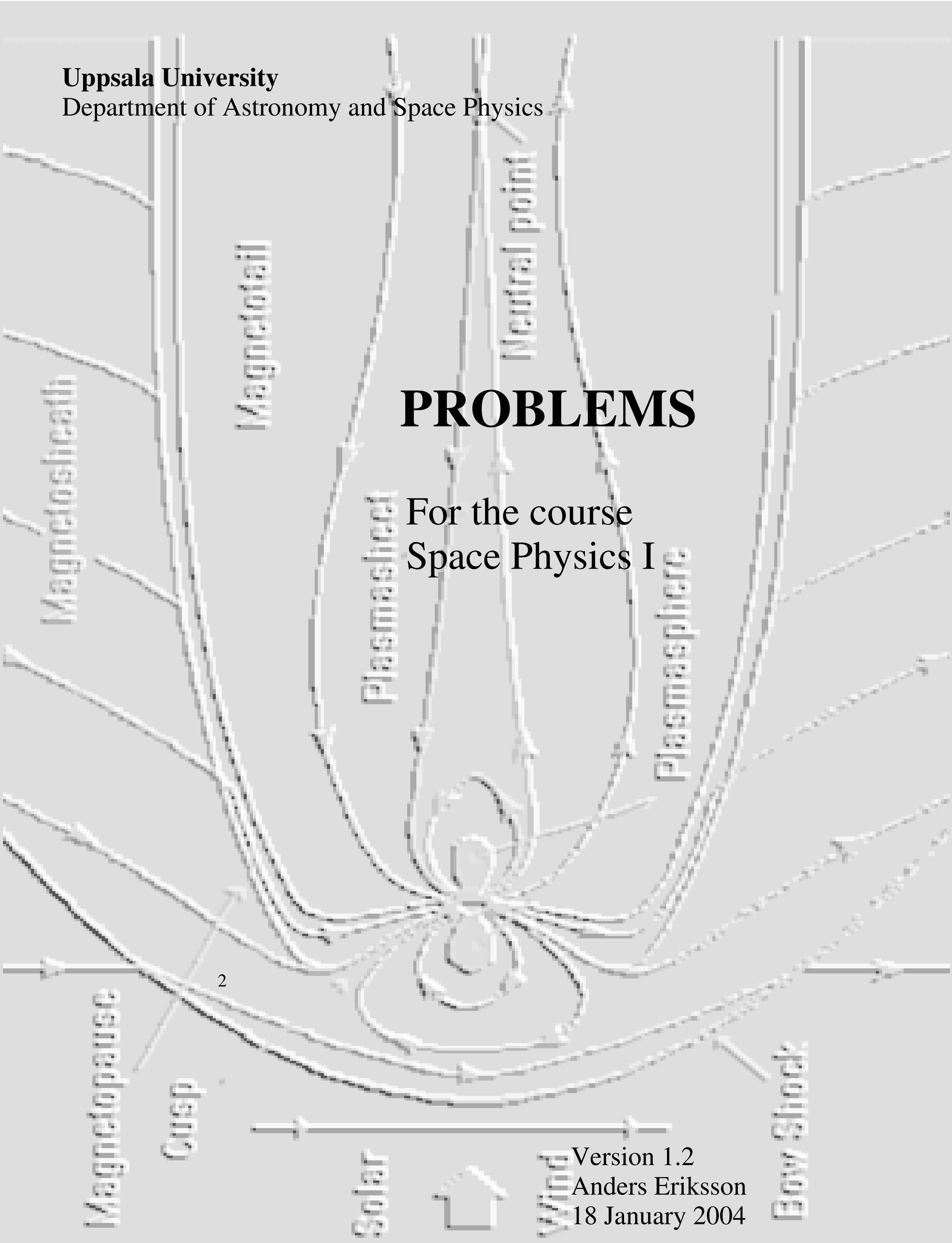
Uppsala University
Department of Astronomy and Space Physics

PROBLEMS

For the course
Space Physics I



Version 1.2
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Contents


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Revision history

1.0  2002-11-11  Initial version
1.1  2002-11-20  References to old books removed,
                 some corrections
1.2  2004-01-18  Some further corrections.
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Problems marked with a star * are considered to be non-standard problems: do not feel depressed if you do not manage to solve these at first (or even second or third) try.

1 Plasmas, the sun and the solar wind

1. *Debye shielding.* The Debye length is the shielding distance in a plasma, the typical distance over which the influence of any single particle is shielded by the adjustment of motion and position of other particles. A derivation of Debye shielding, considering the potential from a point charge, can be found in Rönnmark's Chapter 6. Assuming that the potential goes to zero at infinity, derive similar expressions for the potential from

   (a) a sphere of radius a at potential V

   (b) an infinite plane at potential V

   (c) * an infinite cylinder of radius a at potential V (involves modified Bessel functions)

2. * *Quasineutrality.* Show that in a plasma at temperature T, the relative charge imbalance $|n_{\mathrm{i}} - n_{\mathrm{e}}|/(n_{\mathrm{i}} + n_{\mathrm{e}})$ caused by thermal fluctuations (whose energy is on the order of KT) is on the order of $(\lambda_{\mathrm{D}}/L)^2$ or less, where L is the length scale of the fluctuation.

3. *Particle flux and the continuity equation.*

   (a) Convince yourself that the number of particles per unit time crossing an area A, whose normal is at an angle θ to a flow with velocity $\mathbf{v}$, must be $nvA\cos\theta$.

   (b) Show that the number of particles passing out from a volume V per unit time must be $\oint_{\partial V} n\mathbf{v} \cdot \mathrm{d}\mathbf{S}$, where ∂V is the boundary surface of V.

   (c) By equating this particle loss to the negative of $\mathrm{d}N/\mathrm{d}t$, where N is the total number of particles in V, derive the continuity equation

   $$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0.$$

4. *Continuity equation.*

   (a) From Maxwell's equations, derive the equation of continuity of electric charge,

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

   (b) Show that this leads to that the change of charge Q in a volume V is entirely due to currents flowing through the boundary surface ∂V, i.e. that

   $$\frac{\mathrm{d}Q}{\mathrm{d}t} + \oint_{\partial V} \mathbf{j} \cdot \mathrm{d}\mathbf{S} = 0.$$

5. *Fluid equation of motion.* For a neutral gas in a gravitational field (which was the model from which we derived the existance and properties of the solar wind), we have written the equation of motion as

$$\rho_{\mathrm{m}} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \rho_{\mathrm{m}} \mathbf{g}$$

where p is the pressure and $\mathbf{g}$ is the gravitational field strength (the same thing as the ordinary g but with its direction specified to form a vector). Show that this indeed can be seen as "Newton's second law per unit volume", i.e. that the total force on a volume V is the sum of the pressure on that volume, $-\oint_{\partial V} p\mathrm{d}\mathbf{S}$, and the gravitational force, $m\mathbf{g}$.

6. *Solar mass loss.* From the intensity of solar electromagnetic radiation at Earth's orbit, 1370 W/m^2, and the solar wind speed and density, typically 350 km/s and 5 cm^{-3} (also at Earth's orbit), estimate the time scale for removing the solar mass by conversion of mass to electromagnetic radiation in fusion reactions, and by solar wind mass transport.

7. *Solar wind.* If the solar wind is assumed to have constant speed around 400 km/s, how long time does it take to reach Mercury, Earth, Mars, Jupiter, Saturn, Neptune?

8. *Solar prominence.* A simple model of the magnetic field of a solar prominence is

$$\mathbf{B} = B_0 \left(\hat{\mathbf{x}} \cos kx - \hat{\mathbf{z}} \sin kx \right) \, e^{-kz}.$$

Find the equation for the field lines, and sketch a plot. What is the current distribution $\mathbf{j}(\mathbf{r})$ required to maintain this magnetic field? Is this a phycically possible prominence model in the sense that it does not require magnetic monopoles?

9. * *Frozen-in magnetic field.* Use the equation of motion for charged particle species,

$$nm \frac{d\mathbf{v}}{dt} = nq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla(nKT),$$

to motivate the conditions

$$\left| \frac{\partial}{\partial t} \right| \ll \omega_{\mathrm{c}}$$

$$|\nabla| \ll 1/r_{\mathrm{g}} \text{ and } \frac{qvB}{KT}$$

for the validity of the frozen-in field condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$.

10. *Solar wind electric field.* Estimate the electric field (as seen by an observer in the frame of the Earth) in the solar wind from the solar wind parameters tabulated on page 92 in Kivelson-Russell.

11. *The interplanetary magnetic field.* Assume a spherically symmetric, stationary (not varying with time, $\partial/\partial t = 0$), radially expanding solar wind $\mathbf{v}(\mathbf{r}) = v(r)\,\hat{\mathbf{r}}$, into which is frozen a stationary magnetic field $\mathbf{B}(\mathbf{r})$ from the sun.

    (a) Consider the area element d$\mathbf{S}$ with normal direction along $\hat{\mathbf{r}}$. How will this element depend on r as it flows with the radially expanding solar wind?

    (b) How will the radial component of the frozen-in magnetic field B_r depend on r for any plasma element blowing with the solar wind?

    (c) Consider some plasma element in the solar wind in the equatorial plane. How will the ratio B_ϕ/B_r change with distance r for this plasma element? Also calculate how B_ϕ depends on r.

(d) How does the results you got here compare to the Parker spiral model developed in Section 4.3.2 in Kivelson-Russell?

12. *Interplanetary magnetic field.* If the typical angle of the IMF to the radial direction from the sun at 1 AU (Earth orbit) is $45°$, what is it at Saturn?

13. *Solar wind and IMF.* Assuming that the solar wind expands with constant velocity and at constant temperature, and that the interplanetary magnetic field is described by the expressions for B_r and B_ϕ in Section 4.3.2 of Kivelson-Russell, how will the plasma beta (ratio of thermal to magnetic energy density) and the ratio of kinetic to magnetic energy density change with r?

14. *Solar wind and IMF*. Figure 1 shows 10 minutes of solar wind electric and magnetic field data from one of ESAs four Cluster spacecraft. The coordinate system is known as GSE (geocentric solar ecliptic), where x points to the sun, y is in the ecliptic plane towards dusk (opposite to planetary motion).

    (a) How does the direction of the magnetic field in this period compare to what is expected from the Parker spiral?

    (b) Assuming that the solar wind flows purely in the $-x$ direction, i.e. radially away from the sun, estimate the solar wind speed during this time interval.

    Hint: What should the electric field be in the solar wind rest frame?

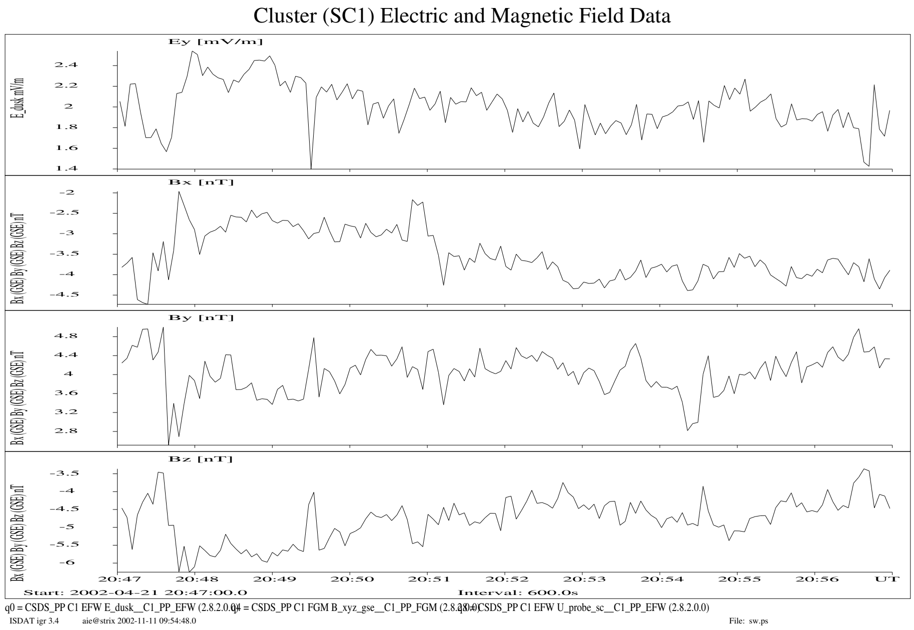

Figure 1: Cluster electric and magnetic field data from the solar wind. E-field data by the Swedish Institute of Space Physics, Uppsala; B-field data courtesy of Imperial College of Science and Technology, London.

15. *Comet tails*. A comet is surrounded by a cloud of gas and dust, evaporated from the comet body by the sunlight. As is the case in planetary atmospheres, this gas is partly ionized, again by the solar radiation. Quite often, comets show two tails. This is attributed to the different acceleration mechanisms operating on neutral particles and ions/electrons.

    (a) The particles in the neutral tail is accelerated in the direction away from the sun mainly by the radiation pressure of the sunlight. For a comet at 1

AU distance from the sun, estimate the force on an oxygen ion due to the radiation pressure.

(b) Charged particles are also affected by electromagnetic effects. For a comet at 1 AU, estimate the electromagnetic force on an oxygen ion. Compare to what you got for the oxygen atom above.

Hints: The radiation pressure p_{rad} is related to the radiation energy flux I_{rad} by $I_{\mathrm{rad}} = p_{\mathrm{rad}}\, c$. Use some table value to estimate the relevant size of an oxygen atom. Remember that the plasma which is created by ionization of comet gases initially has a speed very different from the solar wind – what consequence does this have for the electric field? The initial speed of the comet plasma can be put to zero in a sun-fixed reference system. Use some standard values for the solar wind speed and the interplanetary magnetic field.

16. *MHD forces.* Show that when applied to a conductor of length L carrying a current I perpendicular to a magnetic field B, the MHD force expression $\mathbf{j} \times \mathbf{B}$ results in a force BIL on the conductor.

17. *MHD forces.* Find the equation for the field lines of the field

$$\mathbf{B} = B_0(\hat{\mathbf{x}} + 2x\hat{\mathbf{y}}).$$

What is the magnetic force density at the point (1,0)? What part of this force is magnetic pressure, and what part is magnetic tension?

18. *Pressure balance - sunspot.* Sunspots are dark patches on the sun where the plasma temperature is lower and the magnetic field intensity higher than in the surroundings. If the temperature is 6000 K outside the sunspot and 4000 K inside, the magnetic field is 0.3 T inside the sunspot and negligible outside, and the density is 10^{19} cm^{-3} outside, what is the ratio between the densities inside and outside the sunspot?

2 Magnetospheres and the motion of charged particles

19. *Pressure balance - magnetopause.* The solar wind pressure is dominated by the dynamic pressure $\sim nmv^2$. At a magnetopause, the solar wind pressure is balanced by the pressure inside the magnetosphere. For Earth and Mercury, the magnetic field, approximately dipolar, dominates the magnetospheric pressure. For the solar wind at 1 AU, v and B are given by Kivelson-Russell in page 92, and the strength of the terrestrial magnetic field on the ground at the equator is 30 μT. Mercury has a magnetic dipole moment of $3 \cdot 10^{12}$ Tm^3 and a radius of 2,440 km. Assume that the solar wind speed is the same at Earth and at Mercury. Estimate the standoff distances, in units of the planetary radii, to the magnetopauses of Mercury and Earth.

20. *Magnetic dipole field lines.* Determine the field lines in a dipole magnetic field, and find how the magnetic field intensity varies as a function of r along any particular field line. What is the magnetic field intensity at an altitude of 3000 km for the magnetic field line which leaves the Earth at latitude 65°? At what distance does it cross the equatorial plane?

21. * *Syncrotron radiation.* An accelerated electric charge radiates electromagnetic energy at a rate
$$P = \frac{\mu_0 q^2 \dot{v}^2}{6\pi c(1 - v^2/c^2)^2}$$
where $\dot{v}$ is the acceleration. A gyrating charge in a magnetic field will therefore emit radiation, as the circular gyro motion implies that the particle is continously accelerated. This is called syncrotron radiation. Consider a particle with $v_\perp \neq 0$, $v_\parallel = 0$ in a homogeneous magnetic field B. Derive an equation for how the kinetic energy decreases due to the radiation. Find the half-life time for the kinetic energy of electrons and ions in the terrestrial ionosphere ($B \approx 10\mu$T) and in the solar wind ($B \approx 10$ nT). Estimate the total power radiated from the terrestrial magnetosphere due to syncrotron radiation. Is this energy loss important?

22. *Magnetic moment.* Show that the magnetic flux inside the gyroorbit of a charged particle in a magnetic field is constant if the first adiabatic invariant is conserved.

23. $\mathbf{E} \times \mathbf{B}$ *motion.* A particle of mass m and charge e, initially at rest at the origin, is subjected to constant fields $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$. Derive the orbit $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ of the particle. Plot the motion in the x-y-plane. If done correctly, you should get a curve known as a cycloid. What is the "wavelength" of this curve?

24. *Loss cone.* Calculate the opening angle of the loss cone

    (a) in the equatorial plane

    (b) at 10000 km altitude (Viking satellite)

    (c) at 1700 km altitude (Freja satellite)

    for particles on a (dipolar) magnetic field line which reaches the ground at magnetic latitude 70°.

25. *Adiabatic motion.* The bold spaceman Spiff, famous interplanetary explorer, cruises leisurely with his spaceship at a point P in the magnetosphere of the yet-to-be-discovered planet Zondarglash-B (see Figure 2). Hideously ugly and

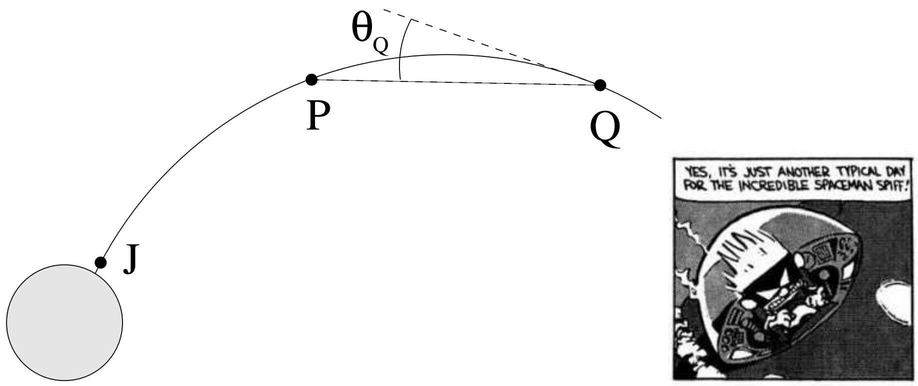

Figure 2: The tense situation when Spiff encounters the space monsters.

extraordinary evil space monsters in another space ship at a point Q at the same magnetic field line as P try to kill Spiff by blasting him with a deadly ray of ionized antimatter. The monsters, who did never pass their course in space physics, act intuitively and fire along the line of sight from Q to P. The ion gyroradius can be considered small compared to the scale length of inhomogeneities in the magnetic field. The magnetic field strength increases monotonically from Q down to J (the planetary ionosphere). The battle may end in three ways:

(a) Triumph of the evil: Spiff is destroyed by the ion ray, monsters survive.

(b) Spiff as well as monsters are killed.

(c) The monsters are fried by the ion ray, while our hero Spiff survives to continue his glorious career.

Explain why these cases arise. Deduce inequalities the angle θ_Q must satisfy for each of these cases. In the drama described above, the actual magnetic field values were $B_Q = 1$ μT, $B_P = 9$ μT, $B_J = 100$ μT, and $\theta_Q = 30°$. What was the outcome of the ferocious battle?

26. * *Bounce motion.* Consider a particle trapped between two magnetic mirrors. Show that

$$v_{\|}(s) = v\sqrt{1 - B(s)/B_{\rm m}}$$

where v is the (constant) speed of the particle, s a coordinate along the magnetic field line, and $B_{\rm m}$ is the magnetic field in the mirror points.

(a) Derive integral expressions for the bounce period, i.e., the time it takes for the particle to go from one mirror to the other and back again.

(b) Write an integral expression for the distance $2\ s_{\rm b}$ which the particle travels along the field line between the mirror points, not including the gyromotion (*Hint:* this is quite trivial, no calculation at all involved).

(c) Near the point where B is minimum, we can Taylor expand the distance in (b) as

$$B(s) = B_0 + as^2$$

if $s = 0$ is the minimum point, B_0 the magnetic field in this point, and a is a constant. Show that particles which have close to 90° pitch angle near the minimum point, and thus cannot travel far from this point, oscilate harmonically along the field line according to

$$\frac{{\rm d}^2 s}{{\rm d}t^2} + \omega_{\rm b}^2 s = 0$$

Figure 3: Idealized geometry of Fermi acceleration of cosmic rays.

where the bounce (angular) frequency $\omega_{\rm b}$ may be written $\omega_{\rm b} = v_{\|0}/s_{\rm m}$.

(*Hint:* As the magnetic field is assumed to be constant in time, we have $\mathrm{d}^2 s/\mathrm{d}t^2 = \mathrm{d}v_{\|}/\mathrm{d}t = v_{\|}\mathrm{d}v_{\|}/\mathrm{d}s$.)

27. *Radiation belts.* The Earth's radiation belts contains magnetically trapped high energy (MeV range) ions (and electrons, which we skip here as they are relativistic), encircling the Earth mainly due to the gradient drift (as opposed to low energy (eV) particles, for which the corotation electric field dominates). Typical geocentric distance of the radiation belt is 2 $\rm R_E$ for protons. Consider 5 MeV protons with equatorial pitch angle close to $90°$. Calculate drift speed and orbital period around the Earth, modelling the geomagnetic field as a dipole field.

28. *Ring current.* Let the geomagnetic field be represented by a dipole field. Consider a plasma in the equatorial plane at 5 $R_{\rm E}$ with density 10 $\rm cm^{-3}$, consisting of 1 eV protons and 10 keV electrons with $90°$ pitch angle in the equatorial plane. Calculate drift speeds, direction of the drift, and drift orbital period around the Earth. Calculate the current density $\mathbf{j}$ for the ring current carried by these particles. If the cross-section area of the current is $1\ R_{\rm E} \times 1\ R_{\rm E}$, what is the total current? Estimate the strength of the magnetic field from this ring current at the center of the Earth. How is this magnetic field it directed - does it enhance or decrease the geomagnetic field? Also calculate drift velocities and resulting current density for the drift due to the gravitational force on the particles. How does this compare to the magnetic gradient drift?

29. *Fermi acceleration of cosmic rays.* A proton in interstellar space is trapped between two magnetic mirrors (see Figure 3). The magnetic field in the mirror points is $B_{\rm m} = 5B_0$, where B_0 is the minimm field strength along the field line between the mirrors. At $t = 0$, the distance between the mirrors is 10^{10} km, but both are moving towards a point between them with a speed of 10 km/s. The initital energy of the proton is 1 keV, and its pitch angle is $45°$. What pitch angle must the proton have to slip out from the mirror configuration? What energy will it have when slipping out? How long time will it take for it to reach this energy and pitch angle?

30. *Betatron acceleration of cosmic rays.* Consider a proton which initially has an energy of 1 keV and pitch angel $90°$ in an interstellar magnetic field of 10 pT.

    (a) If the magnetic field is increased slowly (so that the first adiabatic invariant is conserved) to 100 pT, what happens to the energy of the proton? If it changes, why? What force does work on the particle?

    (b) Assume that when the magnetic field has reached 100 pT, the proton ellastically collides with another particle, so that its energy is conserved but the pitch angle after the collission becomes $45°$. If the magnetic field strength decreases to its initial value 10 pT again after the collission, what will the energy of the proton be?

(c) If the energy of the proton was changed in this procedure, from where did it come or where did it go? Can both the particles (the proton and its collission partner) gain energy?

Can you think of any other context in which the process could be important?

31. *Relativistic motion.* As a rule, relativistic effects become important when the kinetic energy (from non-relativistic theory) becomes comparable to the rest mass of the particle. Calculate these energies for electrons and ions. Will relativistic effects be important for radiation belt particles? Cosmic rays? Derive relativistic expressions for the gyroradius and the cyclotron frequency.

32. *Current sheets.* Thin layers of current are found in many magnetospheric contexts, like the magnetopause current, the cross-tail current, or the field-aligned Birkeland currents. The simplest case is the sheet that is infinite in the y and z directions, has a thickness of $2a$ in the x direction, and carries a homogeneous current of total strength I flowing along $\hat{\mathbf{z}}$. Calculate the magnetic field from such a sheet.

33. *Tail current.* In the magnetospheric tail, the magnetic field may be modelled as homogeneous with a strength of 10 nT. The direction is towards the Earth above the equatorial plane and away from the Earth below.

    (a) Determine the strength and direction of the surface current (unit: A/m) which flows across the tail.

    (b) Describe the motion of 1 keV protons and electrons which at $t = 0$ are in the equatorial plane with velocity pependicular to the equatorial plane and to the magnetic field.

    (c) Show that the motion of the particles represents a current in the same direction as the surface current discussed above.

    (d) Estimate the plasma density which is needed to carry all the surface current if all particles have energy 1 keV and 90° pitch angle.

34. *Magnetotail.* Consider the tail of the magnetosphere to consist of two lobes with oppositely directed magnetic field with strength 10 nT far away from the equatorial plane. Assume that the current sheet between the lobes is not the zero thickness layer considered in the previous problem, but has a thickness of 5000 km in which the current is homogeneously distributed. Calculate the current density $\mathbf{j}$ and the variation of the magnetic field in this sheet, as well as the magnetic force per unit volume ($\mathbf{j} \times \mathbf{B}$). Show that $\mathbf{j} \times \mathbf{B} = -\nabla B^2/(2\mu_0)$. The magnetic force may thus be interpreted as the gradient of a pressure $B^2/(2\mu_0)$ which varies in space. Which forces in the plasma may balance this force? What does this imply for the plasma in the central plasma sheet (the region where the cross-tail current flows)?

35. *Substorms.* In a substorm, the magnetic energy stored in the tail may be dissipated within 10 minutes. Consider the tail in the previous problem. If it is a cylinder of radius and length 10 R_{E}, how much energy is stored in the tail magnetic field? If all this is dissipated in 10 minutes, what is the average power in the substorm ?

36. *Loads and generators in MHD circuits.* In the ideal magnetohydrodynamic (MHD) approximation, the plasma is described by the frozen-in condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ and the equation of motion $mn\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{j} \times \mathbf{B}$. From the two MHD

equations, show that the released electric energy per unit volume and time, $\mathbf{j} \cdot \mathbf{E}$, goes to kinetic energy (per unit volume). Regions in space where $\mathbf{j} \cdot \mathbf{E} > 0$ are therefore "motor" areas where electric energy is converted to plasma motion, and constitute a load on the circuit. Regions where $\mathbf{j} \cdot \mathbf{E} < 0$ are "generators", where kinetic energy is converted to electromagnetic energy.

3 Ionospheres and conductivity

37. * *E region recombination.* In an ionosphere consisting of NO^+ ions and electrons with density 10^{11} m^{-3} (Earth's E region), find the typical lifetime of an NO^+ ion by putting the source term (ionization) to zero. Assume that the loss term is dissociative recombination $NO^+ + e^- \rightarrow N + O$ with reaction constant $\alpha = 3 \cdot 10^{-13}$ m^3/s. *(Answer:* $1/(\alpha n)$ = 30 s.)

38. *F region charge exchange.* Assume that the plasma in the terrestrial F region around 300 km altitude consists of O^+ ions and electrons with number density 10^{12} m^{-3}. In the same manner as above, estimate the typical lifetime of O^+ ions by calculating the time at which the density as decreased to $1/e$ of its initial value. Assume that the loss process is charge exchange $O^+ + N_2 \rightarrow NO^+ + N$ with reaction constant $k = 10^{-18}$ m^3/s. The density of molecular nitrogen may be found from Figure 7.7 in Kivelson-Russell. *(Answer:* $1/(kn_{N_2}) = 10^4$ s.)

39. * *Ionospheric equilibrium density.* In a pure oxygen atmosphere of density n_{O_2}, photoionization creates O^+ and O_2^+ ionis at rates q_1 and q_2, respectively (units: $m^{-3}s^{-1}$). These ions are lost in the reactions $O^+ + O_2 \rightarrow O_2^+ + O$ and $O_2^+ + e^- \rightarrow O + O$ with reaction constants k and α, respectively. Find an expression for the electron number density at equilibrium. *(Answer:* $n_e = \frac{q_1}{2kn_{O_2}} + \sqrt{\frac{q_1+q_2}{\alpha} + \left(\frac{q_1}{2kn_{O_2}}\right)^2}$.)

40. * *Recombination.* Consider what happens if the ionization in the oxygen ionosphere in the previous problem suddenly stops ($q_1 = q_2 = 0$ for $t > 0$). Derive a differential equation for the variation of the electron number density with time. *(Answer:* $\mathrm{d}n_e/\mathrm{d}t + \alpha n_e^2 = \alpha n_e q_1/(kn_{O_2})e^{-kn_{O_2}t}$.)

41. * *F region recombination.* After the charge exchange process described in problen 38, F region electrons are lost by the dissociative recombination process $NO^+ + e^- \rightarrow N + O$, with reaction constant $\alpha = 3 \cdot 10^{-13}$ m^3/s. Estimate the characteristic time for decay of the electron density.

42. *Chapman theory.* Show that the maximum electron number density, as predicted by the Chapman theory, occurs at an altitude where the intensity of the solar radiation (in UV, 10 - 100 nm) has decayed to $1/e$ of the intensity above the atmosphere, and that the maximum electron density is $n_e^{\max} = \sqrt{a_j I_0 \cos\phi/(a_a a_r He)}$. What is the maximum electron density if $I_0 = 3$ mW/m^2, $\phi = 45°$, $H = 10$ km, $a_r = 3 \cdot 10^{-13}$ m^3/s, $a_a = 10^{-21}$ m^2, the ionization energy E_i is 15 eV, and 1/6 of the absorbed energy goes to ionization. *(Answer:* $1.3 \cdot 10^{11}$ m^{-3}, *typical for the E region.)*

43. * *The conductivity tensor.* As you know, there are currents flowing along the magnetic field lines in the auroral regions. The Pedersen conductivity is low in the magnetosphere, so the currents close in the ionosphere. Consider a planet with an ionosphere and a magnetic field. The ionosphere is assumed to reach all the way down to the (insulating) planetary surface. In this particular ionosphere, the Pedersen and Hall conductivities vary with the altitude z above the ground as $\sigma_P = \sigma_0\, e^{-z/a}$ och $\sigma_H = \sigma_0\, e^{-z/b}$. The parallell conductivity can be assumed infinite for all z. The magnetic field is homogeneous and vertical, and the planet is very big, so that its surface may be assumed to be flat and infinite. Introduce a Cartesian coordinate system (x, y, z) with the origin on the planetary surface and $\hat{\mathbf{z}}$ pointing upwards. Due to processes in the planetary magnetosphere, field aligned currents flow along the magnetic field as follows: One current flows down along the field as a thin current sheet at $x = c$, and another current flows

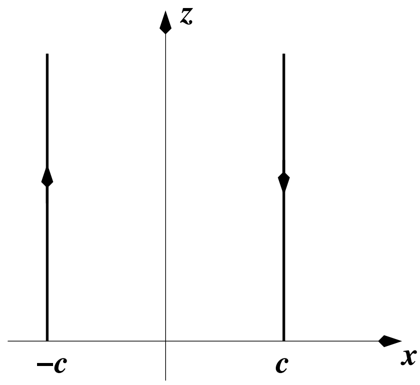

Figure 4: Two current sheets in an ionosphere. The current varies with altitude in the sheets, and closes through currents flowing between the sheets (not shown).

upward in the sheet $x = -c$ (Figure reffig:currsheets). Both current sheets have infinite extent in the y and z directions, and the vertical component of the current density vector may therefore be written as

$$j_z(x, y, z) = K(z)\,[\delta(x + c) - \delta(x - c)]$$

where $K(z)$ is some function with $K(\infty) = K_0$. These two currents close in the ionosphere via a perpendicular current density component $j_x(z)$ between the current sheets.

(a) Calculate $j_x(z)$, $K(z)$, and the electric field E_x between the sheets.

(b) Discuss the existence of currents and electric fields in the y direction.

(c) Calculate the power dissipated by the current per unit length in the y direction (the power dissipation per unit volume is $\mathbf{j} \cdot \mathbf{E}$).

(d) Discuss the applicability of this model on real magnetosphere-ionosphere systems.

4 Rockets and spacecraft

44. *Escape velocity.* Calculate the radial speed a body at the Earth's surface need to acquire in order to overcome gravitation and escape into space. Must a real rocket ever acquire this speed if it is to escape from the Earth?

45. *Solar gravitation.* Using table values for solar mass and size, find the acceleration of gravity and the escape velocity at the solar surface.

46. *Escape velocity.* Calculate the radial speed a body at the Earth's surface need to acquire in order to overcome gravitation and escape into space. Must a real rocket ever acquire this speed if it is to escape from the Earth?

47. *Circular orbits.* A satellite is orbiting the Earth with a period of 4 hours. What is its altitude?

48. *Total energy.* The total energy of a spacecraft in a circular orbit is the sum of its potential and kinetic energy. Derive the total energy of a satellite of mass m orbiting the Earth at geocentric distance r.

49. *Fuel budget.* If the exhaust velocity of a one-stage rocket is 3 km/s, what is the minimum fraction of the rockets mass that needs to be used for fuel if the rocket is to reach a circular orbit at 250 km altitude?

50. *Where to burn your fuel.* Consider a spacecraft passing a planet, i.e. first falling in toward the planet and then going away again, without changing direction in the process (in reality, this is possible only if falling through the centre of the planet – let us assume there is a convenient tunnel). If we have some fuel onboard which we want to use for getting as much extra speed as possible when we have left the planet, where should we fire the engine – when we are close to the planet, or far away?

51. *Planetary flybys.* Close planetary flybys are often used by deep space probes in order to gain momentum. For example, Rosetta will pass Mars once and Earth three times on its nine-year journey to comet Churyomov-Gerasimenko, and Cassini first went in to Venus even though its ultimate goal is Saturn. How is it possible to gain momentum from a planetary flyby, even if one does not fire any engine? Shouldn't the spacecraft come out with just the same kinetic energy, i.e. the same speed, after the flyby as before? How does this work?

52. *Equilibrium temperatures.* Calculate the equilibrium temperatures of the following objects, far from the Earth but still at a distance of 1 AU from the Sun:

    (a) A sphere covered with gold (α/ϵ = 5.50)

    (b) A sphere covered with white paint (α/ϵ = 0.22)

    (c) A cube, with one of its side facing the sun at right angles, covered with white paint

    (d) A thin plate, with one of its side facing the sun at right angles, covered with white paint

53. *Spacecraft heating needs.* Consider a cubelike spacecraft, 2 m on each sides, with one of its sides facing the sun at right angles. The surfaces are assumed to be covered by a material with $\alpha = 0.35$ and $\epsilon = 0.4$.

    (a) What is the equilibrium temperature of the spacecraft at Earth orbit?

    (b) One may raise the temperature by using electric heaters. How much power would be needed to raise the temperature of the spacecraft by $10°$C at Earth orbit?

(c) If the efficiency of the solar panels providing the electricity is 0.35, how large area must they have to provide the power necessary for the heating in (b)?

(d) Now put the same spacecraft out at Jupiter, 5.1 AU from the sun. Recalculate equilibrium temperature, power needed for raising $10°$C and necessary solar panel size for this environment.

54. *Mission to the corona.* To understand the fundamental question of coronal heating, we would very much like to do measurements, particularly of the magnetic field and of the waves and particles in the plasma, inside the corona. The problem of course is that a spacecraft going so close to the sun gets very hot. One way of keeping cool would be to have a cone-shaped spacecraft with the top pointing toward the sun. If the top angle of the cone is small, this gives a much larger emission area (the mantle area plus the bottom area) than absorbtion area (equal to the bottom area). If the spacecraft is to become no hotter than $T_{\max}$, what is the needed top angle as function of distance to the sun and absorption ratio α/ϵ? If $T_{\max}$ = 77°C for a conical spacecraft that should come as close as one solar radius above the Sun's surface and has a surface material with $\alpha/\epsilon = 0.2$, what is the largest top angle allowed?

55. *Plasma heat flux.* The temperature in the solar wind is quite high, typically perhaps 10 eV which is equivalent to 10^5 K. If all the thermal energy of the solar wind particles hitting a spacecraft would transfer their energy completely to thermal energy of the spacecraft body, what energy flux would this correspond to, assuming typical values for solar wind density and velocity (5 cm^{-3} and 400 km/s, respectively)? How does this compare to the solar radiation flux of 1.4 kW/m^2 at 1 AU?

56. *Sailing on sunlight and on solar wind.* The solar wind as well as the radiation pressure of the sunlight give rise to antisunwardly directed forces on bodies in space.

(a) Estimate these forces on a 1 m^2 area in space. Which is the larger one?

(b) Sailing in space using the bigger of these forces has been discussed as a means for going to the far reaches of the solar system. A technical problem is that big lightweight sails are needed. Instead of using complicated mechanical sails, it has been suggested to create an artificial magnetosphere around the spacecraft. Using a pressure balance argument, derive an approximate relation between the magnetic dipole moment on the spacecraft and the resulting force on the spacecraft.

(c) Estimate the solar wind force on the Earth's magnetosphere.

5 Miscellaneous

57. *Large scale forces.* Formally, the gravitational and electric forces has the same dependence on distance ($1/r^2$). Why is the gravitational force important for large scales (planetary or galactic dynamics, for instance) but not the Coulomb force? Find the electric charge on the sun and the Earth which would be necessary to get an electric force as strong as the gravitational force between them. What electric field would this charge give on the ground?

Answers

1. (a) $V\frac{a}{r}\exp((a-r)/\lambda_\mathrm{D})$
   (b) $V\exp(-|x|/\lambda_\mathrm{D})$
   (c) $V\frac{K_0(r/\lambda_\mathrm{D})}{K_0(a/\lambda_\mathrm{D})}$
2. Hint: Thermal fluctuations have energy on the order of KT or less, so the highest potential energy that can be caused by a fluctuation is $e\Phi \sim KT$. Use Gauss' law to find the relative charge imbalance.
3.
4.
5.
6. EM radiation: mass loss $4.3 \cdot 10^9$ kg/s $\Longrightarrow$ loss time $1.5 \cdot 10^{13}$ years
   Solar wind: mass loss $1.4 \cdot 10^9$ kg/s $\Longrightarrow$ loss time $4.4 \cdot 10^{13}$ years
   (assuming 75 % H and 25 % He in solar wind)
7. Approximate numbers: Mercury 40 hours, Earth 4 days, Mars 1 week, Jupiter 3 weeks, Saturn 6 weeks, Neptune 4 months
8. Field lines: $z = -\frac{1}{k}\ln(C\cos kx)$
9.
10. About 3 mV/m
11. (a) As r^2, by purely geometrical considerations. This element occupies a constant solid angle as seen from the sun, i.e. it will always occupy the sam fraction of the sphere of radius r centred on the sun.
    (b) As $1/r^2$, because the flux through the area should be constant (frozen-in condition), and we just found that the area goes like r^2.
    (c) $B_\phi/B_r \propto r/v(r)$. To find this, do a similar analysis as above for an area element with its normal direction in the $\hat{\phi}$ direction: its sides will be $r\mathrm{d}\phi$ and $dr(r) = v(r)\mathrm{d}t$, so that $B_\phi \propto 1/(rv(r))$.
    (d) The treatment you just have done is general, while the Parker spiral essentially ties a boundary condition (solar rotation and B_r dominating at solar surface) to the problem. There is thus no contradiction: you get the same dependence of B_r and B_ϕ as in the Parker spiral, but with an arbitrary value of the ratio B_r/B_ϕ.
12. $84°$, using the preceeding problem.
13.
14. The solar wind is a conducting medium (plasma), so the electric field in the solar wind rest frame should be close to zero: $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. This gives $\mathbf{v}$ from the given $\mathbf{E}$ and $\mathbf{B}$ components, if we assume that the solar wind is flowing radially outward, i.e. that $\mathbf{v} = (v, 0, 0)$ in the given coordinate system.
15. (a) About $7 \cdot 10^{-26}$ N, using an oxygen atomic radius $r = 70$ pm. Remember that the effective atomic area for the solar radiation pressure is πr^2, not $4\pi r^2$, as the sun shines from a particular direction.
16.

17. Field lines: $y = C + x^2$
    $\mathbf{j} \times \mathbf{B}(1,0) = \frac{2B_0^2}{\mu_0}\,(\hat{\mathbf{y}} - 2\,\hat{\mathbf{x}})$
    The first term is due to magnetic tension, the second is due to magnetic pressure.

18. $n_{\mathrm{in}}/n_{\mathrm{out}} \approx 1.4$

19. Earth $\sim 8\ R_{\mathrm{E}}$, Mercury $\sim 1.1\ R_{\mathrm{M}}$
    (Mass conservation requires $n_{\mathrm{SW}} v_{\mathrm{SW}} r^2 = \mathrm{constant}$, so if v_{SW} is assumed the same at Mercury and Earth, n_{SW} must be a factor $1/0.39^2$ higher at Mercury than at Earth)

20. Field line equation: $r = r_0\ \cos^2\theta$
    $B(r) = B_0\left(\frac{R_{\mathrm{P}}}{r}\right)^3\sqrt{4 - 3\,r/r_0}$
    (B_0 = intensity of magnetic field on planetary surface at magnetic equator, R_{P} = planetary radius)
    For the given field line, $B(R_{\mathrm{E}} + 3000\,\mathrm{km}) = 13\ \mu$T, $r_0 = 5.6\ R_{\mathrm{E}}$

21.

22.

23. $\mathbf{r}(t) = \frac{E}{\omega_{\mathrm{c}} B}\left([\omega_{\mathrm{c}} t - \sin\omega_{\mathrm{c}} t]\hat{\mathbf{x}} + [1 - \cos\omega_{\mathrm{c}} t]\hat{\mathbf{y}}\right)$
    "wavelength" = distance between points touching the x axis = $\frac{2\pi E}{\omega_{\mathrm{c}} B}$

24.

25. Our daring hero Spiff survives yet another attack, as you may find out by considering the adiabatic motion of ions starting at point Q: particles with 30° pitch angle can only reach to a point where the magnetic field is a factor $1/\sin^2 30^\circ = 4$ times stronger than at Q.

26. (a) $T_{\mathrm{b}} = \frac{2}{v}\int_{s_{\mathrm{m}}}^{s'_{\mathrm{m}}} \frac{\mathrm{d}s}{\sqrt{1 - \frac{B(s)}{B_{\mathrm{m}}}}}$

27. Protons: $v_{\nabla B} = 300$ km/s, $T_{\mathrm{drift}} = 2$ minutes

28. Gradient-curvature drift:
    Electrons: 1.3 km/s eastward drift, 5 hour drift period
    Ions: 13 cm/s westward drift, 50 year drift period
    $j_{\mathrm{grad+curv}} = 2$ nA/m^2, $I = 80$ kA, $B = 0.08$ pT (adds to the geomagnetic field on ground)
    Gravitational drift:
    Electrons: 0.23 mm/s westward drift
    Ions: 43 cm/s eastward drift
    $j_{\mathrm{grav}} = 0.6$ pA/m^2
    $j_{\mathrm{grav}}/j_{\mathrm{grad+curv}} \approx 1/3000$

29. $\sin^2\alpha < 1/5$, 2.5 keV, ~ 8 years (using the 2nd adiabatic invariant)

30. (a) Energy increases to 10 keV due to emf associated with $\partial\mathbf{B}/\partial t$

(b) 5.5 keV

(c) The emf does work on the particles. Both can gain energy.

31. 0.511 MeV (electrons), 938 MeV (protons)
$\omega_\mathrm{c} = \frac{eB}{\gamma m} = \omega_\mathrm{c}^\mathrm{nonrelativistic}/\gamma$
$r_\mathrm{g} = \frac{v_\perp}{\omega_\mathrm{c}} = \frac{p_\perp}{eB}$

32.

33.

34.

35.

36.

37. $1/(\alpha n) = 30$ s

38. $1/(kn_{\mathrm{N}_2}) = 10^4$ s

39. $n_\mathrm{e} = \frac{q_1}{2kn_{\mathrm{O}_2}} + \sqrt{\frac{q_1+q_2}{\alpha} + \left(\frac{q_1}{2kn_{\mathrm{O}_2}}\right)^2}$

40. $\frac{\mathrm{d}n_\mathrm{e}}{\mathrm{d}t} + \alpha n_\mathrm{e}^2 = \frac{\alpha n_\mathrm{e} q_1}{kn_{\mathrm{O}_2}} \exp(-kn_{\mathrm{O}_2} t)$

41.

42. $1.3 \cdot 10^{11}$ m^{-3}

43. (a) $E_x = -K_0/(a\sigma_0)$
$j_x(z) = -\frac{K_0}{a}\exp(-z/a)$
$K(z) = K_0(1 - \exp(-z/a))$

To find this, we first consider the electric field. As discussed in (b), there is no reason to assume a y-component. As the parallell conductivity is infinite, there can be no electric field in the z direction. Hence the electric field is given by $\mathbf{E}(\mathbf{r}) = E_x(x,z)\hat{\mathbf{x}}$. The situation is static, so the E-field must be curl-free, which gives $\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x} = 0$: thus E_x cannot depend on z. Current continuity ($\nabla \cdot \mathbf{j} = \frac{\partial \rho}{\partial t} = 0$) implies that j_x cannot depend on x between the current sheets, and as $E_x = j_x/\sigma_P(z)$, E_x must also be independent of x. Thus E_x is a constant between the sheets.

Current continuity $\nabla \cdot \mathbf{j} = 0$ also gives that $\mathrm{d}K/\mathrm{d}z = -j_x(z)$ in our case. Using $j_x = \sigma_\mathrm{P} E_x$, we get $K(z) = K_0 + \int_\infty^z j_x(z)\,\mathrm{d}z = K_0 + a\sigma_0 E_x\, exp(-z/a)$; the boundary condition $K(0) = 0$ (no current can flow into the planet) then gives E_x as above, which finally is used to express $j_x(z)$ and $K(z)$.

(b) This is a two-dimensional problem, symmetric in the y direction, so there can be no dependence on the y coordinate. Neither is there any reason to have an E-field in this direction, as there is nothing in the problem stating the direction of this field: we thus assume $E_y = 0$. However, it would be possible to solve the problem for a non-zero E_y, caused by some external process, if such an electric field had been specified in the problem.

As for currents, the non-zero σ_H and E_x will give a Hall current. One should note that the Cowling conductivity is not appliccable to this problem: it applies in a slab of enhanced conductivity, which is not the case here (the plasma is assumed uniform also outside of the sheets).

(c) Dissipated power per unit volume: $\mathbf{j} \cdot \mathbf{E}$.
Integrate over x and z to get the requested $\frac{\mathrm{d}P}{\mathrm{d}y} = \int_0^\infty \int_{-c}^{c} \mathbf{j} \cdot \mathbf{E} \, \mathrm{d}x \, \mathrm{d}z = 2cK_0^2/(a\sigma_0)$.

(d) Current sheets somewhat like the pair given are indeed observed in the auroral regions. In reality, the plasma conductivity would have a more complicated behaviour than in this simple exponential model, it would change particularly in the upward current sheet due to increased ionization by auroral electrons, and the physics of the two sheets would be quite different, but the problem can be considered a first-order model of closure of field-aligned currents in the ionosphere.

44. $v_{\mathrm{esc}} = 11.2$ km/s. This is calculated from equating the needed kinetic energy of the rocket to the change in gravitational potential energy between the Earth's surface and infinity, $Gm_{\mathrm{E}}m_{\mathrm{rocket}}/R_{\mathrm{E}}$. As rockets actually are not burning out when on the ground but are operational up to perhaps 200 km, where the gravitation energy is slightly lower, a little lower speed will actually do, but the difference this makes is quite small, only 0.2 km/s, so the value of 11.2 km/s is essentially true for all launches.

45. $g = 272$ m/s^2, $v_{\mathrm{esc}} = 620$ km/s

46.

47. $h = r - R_{\mathrm{E}}$ = 6439 km

48. $E = -\frac{GMm}{2R}$

49. 92.5 %

50. Wherever a rocket is fired, it will provide the same Δv, given by the rocket equation (assuming short burn time – otherwise $\mathbf{g}$ matters). However, if the velocity varies due to a gravitational field, most work will be done when the velocity is high, because the work done is $\Delta W = mv\Delta v$. Thus it is most efficient firing the engine when v is as high as possible, i.e. close to the planet (which, in this gedankenexperiment situation, means inside the tunnel).

51. Energy is conserved in the system defined by the common centre of mass of the planet and the spacecraft. As the spacecraft is much smaller than the planet, this is almost identically the rest frame of the planet. However, this does not mean that energy is conserved in a reference frame centred on the sun: the planet moves around the sun, so in this system the initial and final kinetic energies, and hence speeds, of the spacecraft may differ. Physically, the spacecraft takes some energy from the planets orbital motion around the sun, but of course the difference to the planet motion is neglegible.

52. (a) 156°C
(b) -81°C
(c)
(d) -45°C

The geometrical design and choice of surface material obviously is important for the resulting temperature of a spacecraft.

53.

54. $\sin\theta/2 < \beta/(1-\beta)$, where $\beta = \frac{\sigma T^4}{\frac{\alpha}{\epsilon} I_0 \left(\frac{R}{R_0^2}\right)^2}$, R is the distance from the sun, R_0 = 1 AU is the Earth's distance to the sun, and I_0 = 1.4 kW/m^2 is the solar irradiation at Earth orbit. For the numbers given, this gives $\theta = 0.02°$. Obviously, this will not be a practical design!

55. $4.8 \cdot 10^{-6}$ W/m^2, i.e. a factor 10^8 below the solar radiation energy flux. This is why interactions with the surrounding medium usually are neglegible outside planetary atmospheres.

56.

57.