Chapter 8 Integral Equations, Moment Method, and Impedances (self and mutual)

Finite Diameter Wires

For thin wire antennas, the current distribution was assumed to be sinusoidal. For finite diameter wires (usually $d > 0.05\lambda$) the sinusoidal current distribution is representative, but not exact. To find the current distribution for a cylindrical antenna, an integral equation is usually derived and solved. Knowing the voltage at the feed terminals and determining the current distribution, the input impedance and the radiation pattern can be obtained.

To avoid mathematical difficulties we restrict our considerations to the linear dipole, but most of the information presented below can be extended to more complicated structures.

Two of the most popular integral equations that are used to determine the current distribution of finite radius wires are Pocklington's integral eq. and Hallén's integral eq.

Hallén's integral equation is restricted to a delta-gap source model, while Pocklington's is more general and works for both delta-gap and magnetic frill generator.

Delta Gap: $\vec{M}_g = -\hat{n} \times \vec{E}^i = \begin{bmatrix} \hat{n} = \hat{\rho} \\ \vec{E}^i = \hat{z}\dfrac{V_i}{\Delta} \end{bmatrix}$

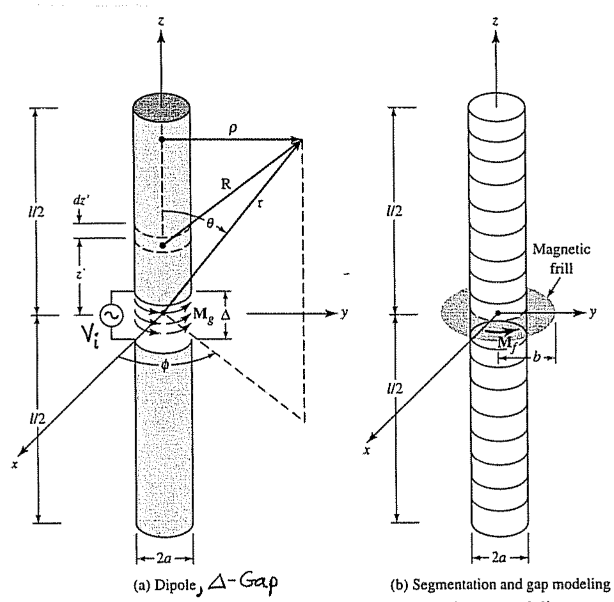
$$= \hat{\phi}\frac{V_i}{\Delta} \qquad -\frac{\Delta}{2} \le z' \le \frac{\Delta}{2}$$

Magnetic Frill Generator: $\vec{E}_f = \hat{\rho}\dfrac{V_i}{2\rho' \ln(b/a)}$, $a \le \rho' \le b$

for TEM mode in coaxial TL

$$\vec{M}_f = -2\hat{n} \times \vec{E}_f = \left[\hat{n} = \hat{z}\right]$$

$$= -\hat{\phi}\frac{V_i}{\rho' \ln(b/a)} \qquad a \le \rho' \le b$$



(a) Dipole, Δ-Gap (b) Segmentation and gap modeling

Figure 8.7 Cylindrical dipole, its segmentation, and gap modeling.

Hallén's Integral Equation

Assume that a finite radius wire of length l is center-fed by an applied delta-gap V_i. If the length of the cylinder is much larger than its radius ($l \gg a$) and its radius is much smaller than the wavelength ($a \ll \lambda$), the effect of the end faces of the cylinder can be neglected.

If the wire has infinite conductivity, the boundary conditions are

1) vanishing tangential $\vec{E}$-fields on the surface of the cylinder.
2) vanishing current at the ends of the cylinder
$$I_z(z')\Big|_{z'=\pm \ell/2} = 0$$

No magnetic current $\Rightarrow$ vector potentials $\vec{F}=\vec{0}$, $\vec{A}\neq\vec{0}$

Taking into account the boundary conditions, the electric field on the surface of the cylinder is

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\epsilon}\nabla_z(\vec{\nabla}\cdot\vec{A}) = 0$$

<u>LHS</u>: the electric current flows on the cylinder along the z-axis and we have $\vec{A}=(0,0,A_z) \Leftrightarrow \vec{J}=\hat{z}J_z$
So we have $\nabla_z(\vec{\nabla}\cdot\vec{A}) = \frac{d^2A_z}{dz^2}$, $k^2 = \frac{\omega^2}{c^2} = \omega^2\mu\epsilon$

<u>RHS</u>: is equal to zero because of vanishing <u>tangential $\vec{E}$-field</u> component on the surface of the cylinder

Our equation becomes (a wave equation)

$$\frac{d^2A_z}{dz^2} + k^2 A_z = 0$$

Assume a symmetrical current density, i.e., $J_z(z') = J_z(-z')$
Then, according to the definition of the vector potential,

$$A_z = \frac{\mu}{4\pi}\int_{-\ell/2}^{\ell/2} I_z(z')\frac{e^{-jkR}}{R}\,dz'$$

A_z is also symmetrical $\left[A_z(z') = A_z(-z'),\ R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]$
This requires a periodic solution of our wave equation:

$$A_z(z) = -j\sqrt{\mu\epsilon}\left[A_1\cos(kz) + B_1\sin(k|z|)\right]$$

Equating the last two expressions and using $\eta = \sqrt{\frac{\mu}{\epsilon}}$, we obtain

$$\boxed{\int_{-\ell/2}^{\ell/2} I(z')\frac{e^{-jkR}}{4\pi R}\,dz' = -\frac{j}{\eta}\left[A_1\cos(kz) + B_1\sin(k|z|)\right]} \qquad (8\text{-}27)$$

which is referred to as <u>Hallén's integral equation</u> for a perfectly conducting wire.

Pocklington's Integral Equation

Now, the boundary conditions ($\vec{E}^{(t)} = 0$ on the surface of the cylinder) are not used directly but are incorporated into the derived equation.

Assume again that $\vec{J} = \hat{z} J_z$ and $\vec{M} = \vec{0}$. The equation for the electric fields is reduced to

$$\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z = j\omega\mu\epsilon E_z^{(s)} \qquad (8\text{-}15)$$

where $E_z^{(s)}$ is the scattered electric field generated by the induced current density $\vec{J}_s = \vec{J}_z$

The expression for the potential $\bar{A} = (0, 0, A_z)$ is given by

$$A_z = \frac{\mu}{4\pi} \int_V J_z(z') \frac{e^{-jkR}}{R} \, dv'$$

Since the current only flows on the surface of the cylinder

$$2\pi a J_z = I_z(z') \quad \Rightarrow \quad J_z = \frac{1}{2\pi a} I_z(z')$$

the volume integral reduces to a surface integral

$$A_z = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \int_0^{2\pi} J_z \frac{e^{-jkR}}{R} a \, d\phi' \, dz' \qquad \rho' = \text{constant} = a$$

$$= \mu \int_{-l/2}^{l/2} I_z(z') \underbrace{\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-jkR}}{4\pi R} d\phi' \right)}_{G(z,z')} dz'$$

where $R = \sqrt{\rho^2 + a^2 - 2\rho a \cos(\phi - \phi') + (z - z')^2}$

Symmetry makes it sufficient to only consider observations at $\phi = 0$

$$R(\rho = a) = \sqrt{4a^2 \sin^2\left(\frac{\phi'}{2}\right) + (z - z')^2}$$

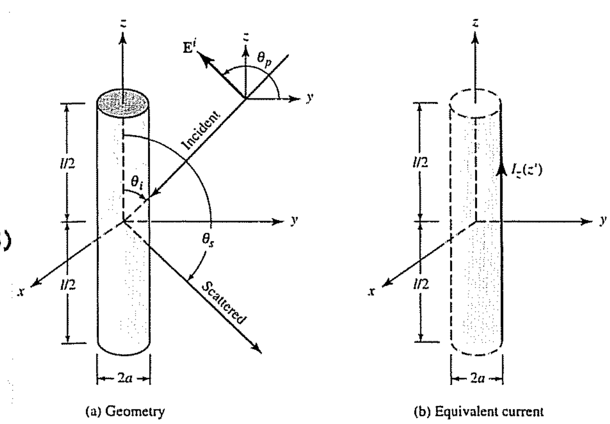
At the surface of the cylinder, the total field must cancel, $E_z^{(t)} = 0$. This means that the incident electric field and the field radiated by the equivalent (filament) current, which is assumed to be located at the center of the wire, $E_z^{(s)}$ must cancel each other

$$E_z^{(t)} = E_z^{(i)}(z) + E_z^{(s)}(z) = 0$$

or

$$E_z^{(i)}(z) = -E_z^{(s)}(z)$$

Inserting A_z into the expression (8-15) for $E_z^{(s)}$ gives for observations on the surface of the cylinder, $\rho = a$



(a) Geometry (b) Equivalent current

Figure 8.5 Uniform plane wave obliquely incident on a conducting wire.

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\int_{-\ell/2}^{\ell/2} I_z(z')G(z,z')\,dz' = j\omega\epsilon E_z^{(s)}(\rho=a) = -j\omega\epsilon E_z^{(i)}(\rho=a)$$

or, by interchanging integration and differentiation

$$\boxed{\int_{-\ell/2}^{\ell/2} I_z(z')\left[\left(\frac{\partial^2}{\partial z^2}+k^2\right)G(z,z')\right]dz' = -j\omega\epsilon E_z^{(i)}(\rho=a)}$$

which is referred to as Pocklington's integrodifferential equation.

For a very thin wire ($a \ll \lambda$)

$$G(z,z') = G(R) \approx \frac{e^{-jkR}}{4\pi R}$$

and

$$\int_{-\ell/2}^{\ell/2} I_z(z')\left[\left(\frac{\partial^2}{\partial z^2}+k^2\right)\frac{e^{-jkR}}{4\pi R}\right]dz' = -j\omega\epsilon E_z^{(i)}(\rho=a)$$

Moment Method Solution

Hallén's integral eq. and Pocklington's integral eq. can both be written in the form

$$\boxed{F(g) = h}$$

F is a known linear integral operator
h is a known excitation function
g is the response function.

The objective is to determine g once F and h are specified.

For Hallén's integral eq, the specification is as follows

$$F \equiv \int_{-\ell/2}^{\ell/2} dz'$$

$$h \equiv -\frac{j}{\eta}\left[A_1\cos(kz) + B_1\sin(k|z|)\right]$$

$$g(z') \equiv g = I_z(z')\frac{e^{-jkR(z')}}{4\pi R(z')}$$

So, we have

$$F(g) \equiv F\{g(z')\} \equiv \int_{-\ell/2}^{\ell/2} I_z(z')\frac{e^{-jkR}}{4\pi R}\,dz' = h$$

We now assume that $I_z(z')$ corresponds to an equivalent current ≡ filamentary line source current localized along the center axis of the wire, so that

$$R = \sqrt{x^2+y^2+(z-z')^2} \qquad (x'=y'=0)$$

Let the observation point be located on the radial distance $\rho = a$ from the wire $\equiv$ filament current.
Then we have

$$R = \sqrt{a^2+(z-z')^2} \equiv R(z,z')$$

We can now write

$$F(g) = F\{g(z,z')\} = \int_0^{\ell} g(z,z')\,dz' = h$$

where we have changed the limits of integration by moving the origin from $z=0$ to $z=-\ell/2$.

The moment method requires that the unknown response function $g(z,z')$ is expanded as a linear combination of N terms and written as

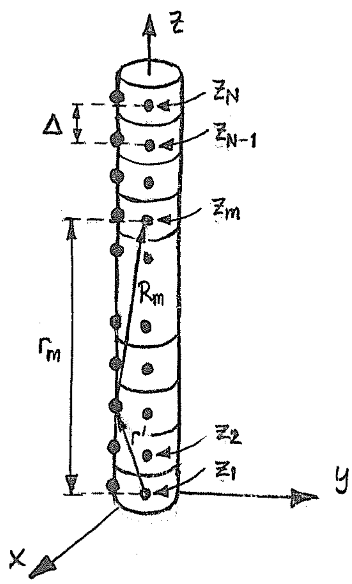
$$g = a_1 g_1 + a_2 g_2 + \ldots + a_N g_N \equiv \sum_{n=1}^{N} a_n g_n(z,z')$$

Substituting this into the integral eq., we have

$$h = \int_0^{\ell} dz' \sum_{n=1}^{N} a_n g_n(z,z') = \sum_{n=1}^{N} a_n \int_0^{\ell} g_n(z,z')\,dz' = \sum_{n=1}^{N} a_n F(g_n)$$

Here a_n is unknown constants and $g_n(z,z')$ are known functions referred to as basis or expansion functions

The basis functions g_n are chosen so that each $F(g_n)$ can be evaluated preferably in closed form, or at the very least numerically. Once the functions g_n are chosen, the only task remaining then is to find the unknown constants a_n



In order to find these constants, we now will divide the wire into N uniform segments each of length $\Delta = \ell/N$, see the Figure.

The next step is to choose the basis functions. To avoid complexity, we choose g_n to be constant over one segment and zero elsewhere

$$g_n(z') = \begin{cases} 1 & z'_{n-1} \le z' \le z'_n \\ 0 & \text{elsewhere} \end{cases}$$

We now choose some fixed point on the surface located on the distance $\rho = a$ from the filament current. So the coordinates of this observation point will be $\rho = a$, $z = z_m$. The integral eq. can be written in the form

$$h = \underbrace{a_1}_{\text{unknown}} \int_0^{\Delta} g_1(z_m,z')\,dz' + a_2 \int_{\Delta}^{2\Delta} g_2(z_m,z')\,dz' + \ldots + a_N \int_{(N-1)\Delta}^{\ell} g_N(z_m,z')\,dz'$$

This is one equation for N amplitude constants c_n, but N linearly independent equations are required. These N equations may be produced by choosing N observation points z_m on the surface of the wire each at the center of each Δ length element. This gives one equation for each observation point. For N such points we have

$$h = a_1 \int_0^{\Delta} g_1(z_1, z')\,dz' + \dots + a_N \int_{(N-1)\Delta}^{\ell} g_N(z_1, z')\,dz' \qquad \text{eq } 1$$

$$h = a_1 \int_0^{\Delta} g_1(z_2, z')\,dz' + \dots + a_N \int_{(N-1)\Delta}^{\ell} g_N(z_2, z')\,dz'$$

$$\vdots$$

$$h = a_1 \int_0^{\Delta} g_1(z_N, z')\,dz' + \dots + a_N \int_{(N-1)\Delta}^{\ell} g_N(z_N, z')\,dz' \qquad \text{eq } N$$

This set of N equations may be written in matrix form as

$$[h_m] = [F_{mn}][a_n]$$

where the h_m column matrix has all terms equal to h,

$$[F_{mn}] = \int_0^{\ell} g_n(z_m, z')\,dz' = \int_{(n-1)\Delta}^{\ell} 1 \cdot dz'$$

and $[a_n] = a_n$ are values of unknown current distribution coefficients. Solving the matrix equation for $[a_n]$ gives

$$[a_n] = [F_{mn}]^{-1}[h_m]$$

Having the amplitude constants a_n, we can determine the unknown current distribution $I(z')$

8.5 Self Impedance

The impedance of an antenna depends on many factors, including its frequency of operation, its method of excitation, and its proximity to the surrounding objects.

The real and imaginary part of the impedance can be found by using the Integral Equation-Moment Method or the Induced EMF Method

8.5.1 Integral Equation-Moment Method:

Hallén's or Pocklington's integral eq. give the current distribution.

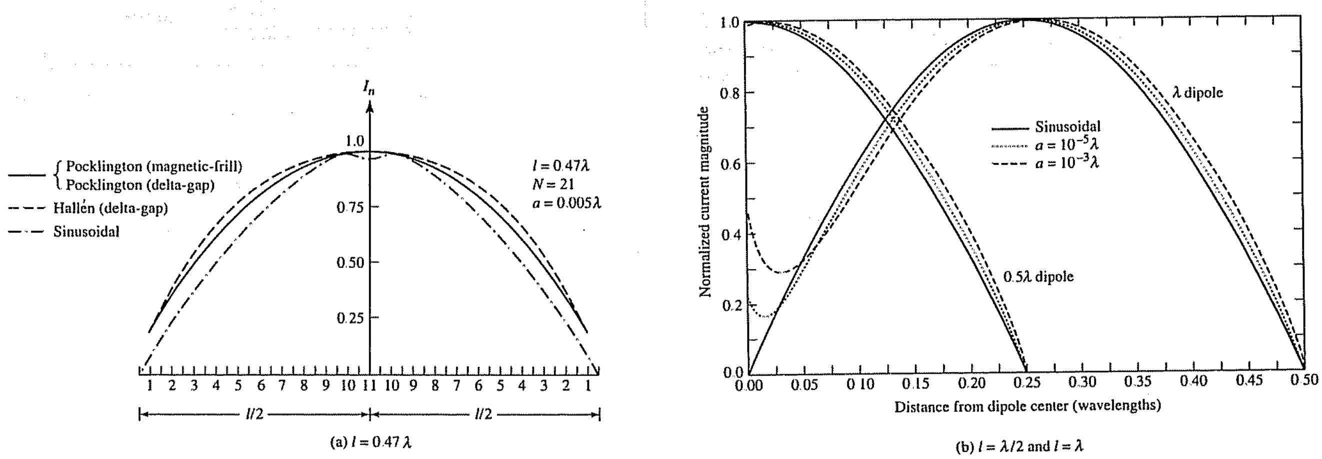


Figure 8.13 Current distribution on a dipole antenna.

The largest deviation from a sinusoidal current distribution is for the feed point of a λ dipole. (Compare with the current distribution for a λ dipole measured during the lab)

When the current is known, the impedance is given by

$$Z_{in} = \frac{V_{in}}{I_{in}}$$

8.5.2 Induced EMF Method

Requires the near-field because otherwise the reactance, which dominates in the near-field region, cannot be determined.

After long derivations, one obtains

$$R_r = R_m = \frac{\eta}{2\pi}\Big\{C + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)\big[S_i(2kl) - 2S_i(kl)\big] + \tfrac{1}{2}\cos(kl)\Big[C + \ln\Big(\frac{kl}{2}\Big) + C_i(2kl) - 2C_i(kl)\Big]\Big\}$$

$$X_m = \frac{\eta}{4\pi}\Big\{2S_i(kl) + \cos(kl)\big[2S_i(kl) - S_i(2kl)\big] - \sin(kl)\Big[2C_i(kl) - C_i(2kl) - C_i\Big(\frac{2ka^2}{l}\Big)\Big]\Big\}$$

where C = Eulers constant = 0.5772,

$S_i(x)$ = the sine integral = $\int_0^x \frac{\sin y}{y} dy$

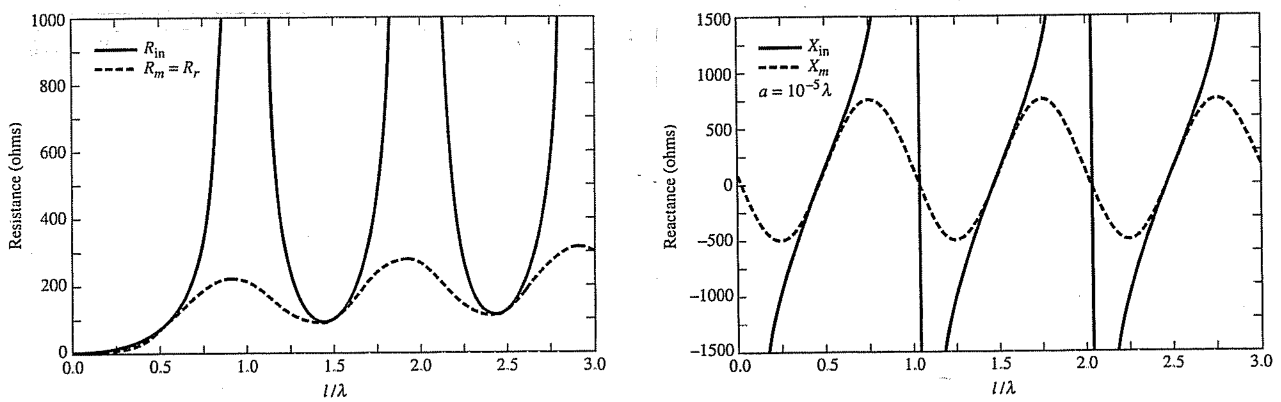
$C_i(x)$ = the cosine integral = $\int_\infty^x \frac{\cos y}{y} dy$

See Appendix III

Input impedance refers to the current at the input terminals and are given by

$$R_{in} = \left(\frac{I_o}{I_{in}}\right)^2 R_r = \frac{R_r}{\sin^2(kl/2)}$$

$$X_{in} = \left(\frac{I_o}{I_{in}}\right)^2 X_m = \frac{X_m}{\sin^2(kl/2)}$$



(a) Resistance

(b) Reactance

Figure 8.16 Self-resistance and self-reactance of dipole antenna with wire radius of $10^{-5}\lambda$.

Effects of finite radii
- small effect for the resistance
- more significant for the reactance

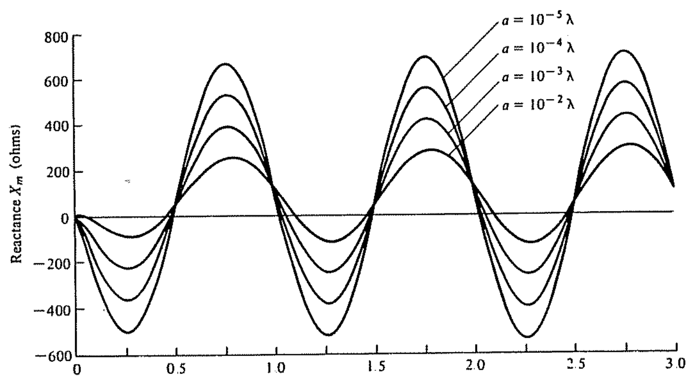


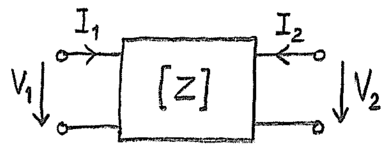
Figure 8.17 Reactance (referred to the current maximum) of linear dipole with sinusoidal current distribution for different wire radii.

8.6 Mutual Impedance Between Linear Elements

In the presence of an obstacle, the current distribution of an antenna will change. When the current distribution changes, the radiated fields and the input impedance changes as well.

The obstacle can, for example, be the ground or another element. Examples of the latter are the parasitic elements of an Yagi-Uda antenna. These elements do not have any current excitation, but there can be a substantial current induced from another source.

For an antenna system of two elements we can represent the system by a two-port by the voltage-current relations



$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2 \qquad (8\text{-}63)$$

where $Z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2=0}$ and $Z_{22} = \left.\dfrac{V_2}{I_2}\right|_{I_2=0}$ are self impedances

and $Z_{12} = \left.\dfrac{V_1}{I_2}\right|_{I_1=0}$ and $Z_{21} = \left.\dfrac{V_2}{I_1}\right|_{I_2=0}$ are mutual impedances

(8-63) can also be written

$$Z_{1d} = \frac{V_1}{I_1} = Z_{11} + Z_{12}\left(\frac{I_2}{I_1}\right) = \text{driving point impedance of antenna \#1}$$

$$Z_{2d} = \frac{V_2}{I_2} = \underbrace{Z_{22}}_{\text{self impedance}} + \underbrace{Z_{21}}_{\text{mutual impedance}}\underbrace{\left(\frac{I_1}{I_2}\right)}_{\text{current ratio}} = \text{driving point impedance of antenna \#2}$$

Two methods to obtain the mutual impedance
1, Integral Equation-Moment Method
2, Induced EMF Method
(the same two as for the self impedance)

8.6.1 Integral Equation-Moment Method

A, Numerical Electromagnetic Code (NEC)

Computes: induced currents and charges, near- and far-zone electric and magnetic fields, radar cross section, impedances or admittances, gain and directivity, power budget, and antenna-to-antenna coupling.

B, Mini-Numerical Electromagnetic Code (MININEC)
Computes: currents, and near-and far fields patterns
It also optimizes the feed excitation voltages that yield the desired radiation pattern.

8.6.2 Induced EMF Method

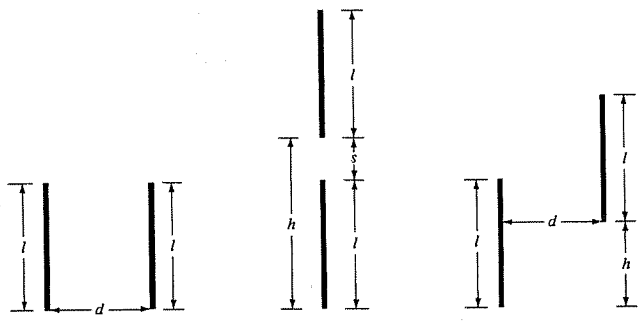


Figure 8.20 Dipole configuration of two identical elements for mutual impedance computations.

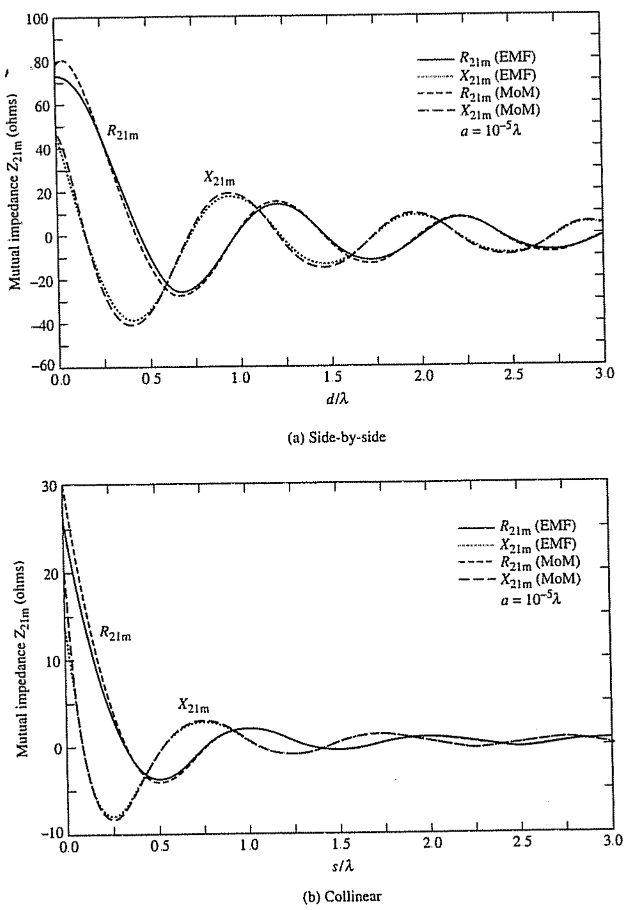


Figure 8.21 Mutual impedance of two side-by-side and collinear $\lambda/2$ dipoles using the moment method and induced emf method.

Two collinear elements influence each other less than two elements placed side-by-side

The mutual impedance is quite insensitive to variations of the wire radius, see Figure 8.22 in Balanis' book.