Solutions to
"Tentamen för Rymdfysik I/MN1/NV1"
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Uppsala universitet
Institutionen för astronomi och rymdfysik
Anders Eriksson

1. (a) The ionized gas (plasma) always streaming out from the sun, essentially radially, at a rate of around 10^9 kg/s.

   (b) No, because the charged particles constituting the solar wind plasma are deflected by the Earth's magnetic field. In the interaction between the solar wind and the geomagnetic field, the latter becomes confined to a region known as the magnetosphere, into which only small amounts of solar wind plasma can penetrate.

   (c) When the magnetic field is frozen into the plasma, the plasma and the magnetic flux moves together in the sense that two physical volumes of plasma who at some time are connected by a magnetic field line will always be connected by a magnetic field line. This makes it possible (but by no means necessary) to interpret magnetic field lines as physical entities moving with the plasma. The concept is appliccable when the electric field in the plasma rest frame is zero, i.e. when $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$, which will be the case for processes long on timescales compared to the ion gyroperiod and on spatial scales compared to the ion gyroradius.

   (d) In order to overcome gravity. This is easy to see by physical common sense: not even a fireworks rocket would get off the ground if you burned the powder in it very slowly during an hour or so – you need to burn it quicly during a few seconds to get an an upward force exceeding mg. More strictly, we can say that this is because of the term $-gt_{\mathrm{burn}}$ in the rocket equation, which will decrease the amount of Δv we get propotionally to the burn time.

   (e) Consider a magnetic mirror, in which the maximum field strength is B_{m}. Assume that we sit at a position on a magnetic field line connected to the mirror, and that the magnetic field strength at our location is B. Particles moving in the direction of the magnetic mirror will mirror and come back to us if they at our position have pitch angle $\alpha > \alpha_{\mathrm{c}}$ where $\sin^2 \alpha_{\mathrm{c}} = B/B_{\mathrm{m}}$. If not, they will slip through the mirror and not come back to us, which means that there will be no particles with $\alpha > 180° - \alpha_{\mathrm{c}}$. This empty cone in velocity space is the loss cone.

   (f) Basically because of the form of the Lorentz force, $\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, which implies that particles can easily flow along the magnetic field, but feels the $\mathbf{v} \times \mathbf{B}$ force when moving perpendicular to it. It is therefore simpler to move particles along field lines than perpendicular to them, resulting in elongated structures.

   (g) The radiation belts are regions of trapped high-energy particles around the Earth or any other planet with a dipole-like magnetic field. Here "high-energy" means energy sufficiently high that the ∇B-drift can dominate over the corotation $\mathbf{E} \times \mathbf{B}$ drift, so the radiation belts can exist outside of the limit where cold plasma is trapped (the plasmasphere). See also Figure 1.

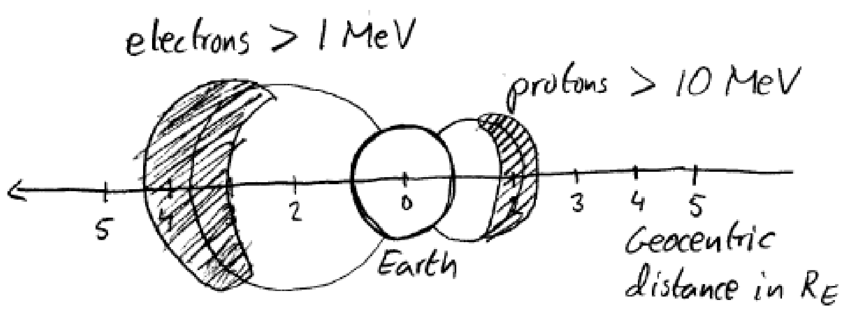
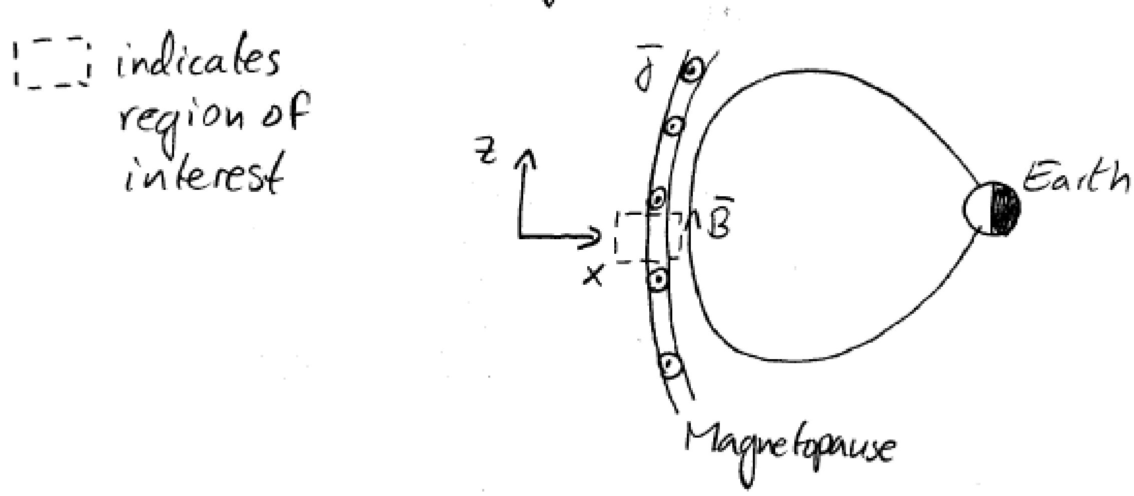


Figure 1: The Earth's radiation belts. Typical locations of the electron belt and the ion belt are shown as shaded areas at left and right, respectively, but of course both of them goes all the way around the Earth. *Note: This figure is more detailed than really necessary for an exam.*

Geometry as in the figure:



(a) Magnetopause current density:

$$\bar{j} = -j\,\hat{y}$$

Magnetic field in region of interest:

$$\bar{B} = B(x,z)\,\hat{z}$$

where we assume

$$B(x,z) = 0$$

for x outside the magnetopause.

Ampère's law:

$$\bar{j} = \frac{1}{\mu_0}\nabla\times\bar{B} = -\frac{1}{\mu_0}\hat{y}\frac{\partial B_z}{\partial x} = -j\hat{y}$$

$$j = \frac{1}{\mu_0}\frac{\partial B_z}{\partial x} \approx \frac{1}{\mu_0}\frac{\Delta B}{\Delta x}$$

where we get $\Delta B \sim 25\,nT$ and $\Delta x \sim 1000\,km$ from the figure.

$$\Rightarrow j \approx \frac{1}{4\pi\cdot 10^{-7}}\;\frac{25\cdot 10^{-9}}{10^{6}}\;A/m^2 \approx \underline{\underline{20\,nA/m^2}}$$

(b) A typical size of the Earth's magnetosphere is given by the stand-off distance (calculated in (c) below), which typically is of the order of 10 R_E. We thus integrate the 20 nA/m^2 magnetopause current over a region which is 1000 km thick in the x-direction and spans around 10 R_E in the z-direction:

$$I \sim 20 \cdot 10^{-9} \cdot 10^{6} \cdot 10 \cdot 6{,}37 \cdot 10^{6}\ \text{A} \sim \underline{\underline{10^5\ \text{A}}}$$

(c) Assuming a dipole field, all we have to do is to find at what distance this field has a strength of 25 nT in the equatorial plane. This is very simple, as

$$B = B_0 \left(\frac{R_E}{r}\right)^3$$

in the equatorial plane. With $B_0 = 30\ \mu$T on the ground, we get

$$r = R_E \left(\frac{B_0}{B}\right)^{1/3} = \left(\frac{30 \cdot 10^{-6}}{25 \cdot 10^{-9}}\right)^{1/3} R_E \sim \underline{\underline{10\ R_E}}$$

(d) Assuming a balance between magnetospheric magnetic pressure and solar wind (or magnetosheath) dynamic pressure, we get

$$\frac{1}{2\mu_0} B^2 \sim (m_p n_p + m_{He} n_{He}) V_{sw}^2 \approx 1.75\, m_p n_e V_{sw}^2$$

assuming $n_p = 0.75\, n_e$ and $n_{He} = 0.25\, n_e$ for the solar wind. We get

$$n_e \sim \frac{B^2}{3.5 \mu_0 m_p V_{sw}^2} \sim \frac{25^2 \cdot 10^{-18}}{3.5 \cdot 4\pi \cdot 10^{-7} \cdot 1{,}67 \cdot 10^{-27} \cdot 4 \cdot 10^{10}}\ \text{m}^{-3} \sim$$

3. (a) The major reason for doing planetary flybys is to gain speed. The idea with trajectories with several planetary swing-bys is to gain the speed needed to complete the mission, in this case to gain sufficient speed to catch up with a comet.

The swing-by mechanism works as follows: in the rest frame of the planet we fly by, the speed of the spacecraft is the same after as before the swing by, but as the direction of the velocity changes, the speed in the rest frame of the sun will be different before and after a flyby. This does not violate conservation of energy in any frame: in the reference frame of the sun, the planet will in fact lose some kinetic energy to the spacecraft.

The main reason for also including Venus in the planning was to gain more possibilities: having three planets to play with (Earth, Venus, Mars) of course is a great gain compared to only two (Mars and Earth – Jupiter and planets further out are too far away). *(Note: it turned out that it is not possible to use Venus flybys for Rosetta: as it is designed, the spacecraft simply becomes too hot when going in to Venus orbit.)*

(b) As there is no conduction to or from the probe, the sunlight power absorbed by the probe,

$$P_{\mathrm{a}} = A_{\mathrm{a}} \alpha I_{\mathrm{rad}}, \tag{1}$$

must in equilibrium be equal to the emitted thermal radiation power,

$$P_{\mathrm{e}} = A_{\mathrm{e}} \epsilon \sigma T^4, \tag{2}$$

so that

$$T = \left(\frac{\alpha}{\epsilon} \frac{A_{\mathrm{a}}}{A_{\mathrm{e}}} \frac{I_{\mathrm{rad}}}{\sigma} \right)^{1/4}. \tag{3}$$

The first fraction in the parenthesis is given by the constants provided in the problem. To find the area ratio $A_{\mathrm{a}}/A_{\mathrm{e}}$, we first note that from the sun, the probe will look like a circle, so

$$A_{\mathrm{a}} = \pi r^2. \tag{4}$$

The emission area A_{e} is the full area of the sphere, i.e.

$$A_{\mathrm{e}} = 4\pi r^2, \tag{5}$$

so that

$$\frac{A_{\mathrm{a}}}{A_{\mathrm{e}}} = 1/4. \tag{6}$$

To get the last unknown parameter in equation (3), the solar radiation intensity at distance R from the sun, we divide the total power of the sun (which Physics Handbook gives as $P_{\mathrm{rad}} = 3.92 \cdot 10^{26}$ W) by the area of a sphere of radius R, i.e.

$$I_{\mathrm{rad}} = \frac{P_{\mathrm{rad}}}{4\pi R^2}. \tag{7}$$

Hence,

$$T = \left(\frac{\alpha}{4\epsilon} \frac{P_{\mathrm{rad}}}{4\pi R^2 \sigma} \right)^{1/4}, \tag{8}$$

yielding, at Earth orbit,

$$T_{\mathrm{Earth}} = \left(\frac{0.47 \cdot 3.92 \cdot 10^{26}}{0.1 \cdot 4\pi \left(1.5 \cdot 10^{11}\right)^2 \cdot 5.67 \cdot 10^{-8}} \right)^{1/4} \approx 412 \text{ K} \approx 139^\circ\text{C} \tag{9}$$

and a factor $1/\sqrt{0.72}$ higher absolute temperature at Venus, i.e.

$$T_{\mathrm{Venus}} = 485 \text{ K} \approx 212^\circ\text{C}. \tag{10}$$

(a) The only force on the particle is the Lorentz force, so the equation of motion is

$$m \frac{d\bar{v}}{dt} = q \bar{v} \times \bar{B}$$

The time rate of change of the kinetic energy then is

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = \frac{d}{dt}\left(\frac{1}{2} m \bar{v} \cdot \bar{v}\right) = m \bar{v} \cdot \frac{d\bar{v}}{dt} =$$

$$= \bar{v} \cdot q \left(\bar{v} \times \bar{B}\right) = 0$$

QED

(b) We can consider the particle as a magnetic dipole μ, which means we do not have to take care of the gyromotion. In this description, the only force on the particle is

$$\bar{F} = -\mu \nabla B. \qquad (3)$$

For a dipole field

$$\bar{B} = -B_0 \left(\frac{R}{r}\right)^3 \left(2 \hat{r} \cos\theta + \hat{\theta} \sin\theta\right), \qquad (4)$$

we get

$$B = B_0 \left(\frac{R}{r}\right)^3 \sqrt{4 \cos^2\theta + \sin^2\theta} =$$

4.

$$= B_0 \left(\frac{R}{r}\right)^3 \sqrt{4 - 3 \sin^2\theta} \qquad (5)$$

It is clear that this expression has a minimum in the equatorial plane ($\theta = 90°$), so here we must have $\frac{\partial B}{\partial \theta} = 0$ and hence

$$\nabla B \big|_{\theta=90°} \parallel \hat{r}. \tag{6}$$

On the other hand, (4) gives for $\theta = 90°$ that

$$\bar{B}(r, 90°) = -B_0 \left(\frac{R}{r}\right)^3 \hat{\theta} = -B(r, 90°)\hat{\theta} \tag{7}$$

so combining (6) & (7), we see there is no force along $\bar{B}$. The particle will thus stay in the equatorial plane forever.

Its motion here will be given by the drift

$$\bar{v} = \frac{1}{q} \frac{\bar{F} \times \bar{B}}{B^2} = \frac{\mu}{q} \frac{\bar{B} \times \nabla B}{B^2} \tag{8}$$

As there will be no θ-component of $\nabla B \big|_{\theta=90°}$ according to (6), we have from (5) that

$$\nabla B \big|_{\theta=90°} = \hat{r} \frac{\partial B}{\partial r}\bigg|_{\theta=90°} = -3B_0 \frac{R^3}{r^4} \sqrt{4 - 3\sin^2\theta} \,\bigg|_{\theta=90°} \hat{r} =$$

$$= -\frac{3}{r} B_0 \left(\frac{R^3}{r^4}\right)\hat{r} = -\frac{3}{r} B(r, 90°)\,\hat{r} \tag{9}$$

(7) & (9) $\Rightarrow$

$$\bar{B} \times \nabla B = -B\hat{\theta} \times \left(-\frac{3}{r} B \hat{r}\right) = \frac{3B^2}{r} \hat{\theta} \times \hat{r} =$$

$$= -\frac{3B^2}{r} \hat{\varphi} \tag{10}$$

(10) in (8) $\Rightarrow$

$$\bar{v} = -\frac{3\mu}{qr}\hat{\varphi} \qquad (11)$$

Now

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{K_\perp}{B} \qquad (12)$$

$$\Rightarrow \quad \bar{v} = -\frac{3K_\perp}{qrB}\hat{\varphi} = -\frac{3K_\perp}{qrB_0}\left(\frac{r}{R}\right)^3\hat{\varphi} =$$

$$= -\frac{3K_\perp}{qRB_0}\left(\frac{r}{R}\right)^2\hat{\varphi} \qquad (13)$$

where $K_\perp$ is the perpendicular kinetic energy of the particle, given as 1 keV.

The time to complete a full orbit will be

$$T = \frac{2\pi r}{v} = \frac{2\pi r q R B_0}{3K_\perp}\left(\frac{R}{r}\right)^2 =$$

$$= \frac{2\pi R^2 q B_0}{3K_\perp}\left(\frac{R}{r}\right) \qquad (14)$$

which in our case becomes $(r = 4R)$

$$T = \frac{2\pi(6371{,}2\cdot 10^3)^2\cdot \cancel{1{,}6\cdot 10^{-19}}\cdot 30\cdot 10^{-6}}{3\cdot 10^3\cdot 1{,}6\cdot 10^{-19}\cdot 4}\ \mathrm{s} =$$

$$= \frac{\pi}{2}\cdot 6{,}37^2\cdot 10^4\ \mathrm{s} \approx 6.4\cdot 10^5\ \mathrm{s}$$

$$\underline{\underline{\approx 1\ \text{week}}} \qquad (15)$$

5. (a) Equation of motion of a neutral gas in a constant gravitational field:

$$mn\frac{d\bar{v}}{dt} = -\nabla p + mn\bar{g} \qquad (1)$$

Assume static equilibrium: $\bar{v}=0$, $\frac{d}{dt}=0$

Assume ideal gas: $p = nKT$

Assume constant temperature: $\nabla p = KT\nabla n$

Assume horizontal stratification: $\bar{g} = -g\hat{z}$

$$\nabla = \hat{z}\frac{d}{dz}$$

(1) then becomes

$$0 = -\hat{z}\,KT\frac{dn}{dz} - mng\hat{z} \qquad (2)$$

$$\frac{dn}{n} = -\frac{mg}{KT}dz \qquad (3)$$

$$\ln n = C - \frac{mg}{KT}z \qquad (4)$$

$$n = e^{C}\exp\left(-\frac{mgz}{KT}\right) \qquad (5)$$

Denoting the density on the ground ($z=0$) by n_0,

$$n = n_0\exp\left(-\frac{mgz}{KT}\right) \qquad (6)$$

This shows the desired features:

— Exponential density decrease with altitude

— That heavier species decreases faster with altitude.

(b) To get any electron density, we must have some atoms or molecules to ionize, and some radiation to do the ionization. Far out in space, there is a lot of radiation but almost no neutral gas to ionize: hence the upper boundary condition is $n(\infty) = 0$. Down at Earth, we have a lot of air molecules to ionize, but the ionizing part of the solar spectrum has mostly been used up at higher altitudes: hence the other boundary condition $n(0) = 0$. In between, at a few 100 km, the electron density has at least one peak: the ionosphere.

(c) Only bodies with an atmosphere can have ionospheres. Examples of bodies with ionospheres thus include Earth, Venus and Jupiter, while our moon is a good example of a body lacking an ionosphere.