Solution to Exam in Antenna Theory

Time: 18 March 2010, at 8.00–13.00.
Location: Polacksbacken, Skrivsal

You may bring: Laboratory reports, pocket calculator, English dictionary, Råde-Westergren: "Beta",
Nordling-Österman: "Physics Handbook", or comparable handbooks.

Six problems, maximum five points each, for a total maximum of 30 points.

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1. a) If the complex electric field is denoted $\mathcal{E}(\mathbf{r})$, find the corresponding instantaneous (time-dependent) electric field $\mathcal{E}(\mathbf{r},t)$. (1p)

   b) The array factor of a N-element uniform array can be written

$$AF = \frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)},$$

   where $\psi = kd\cos\theta + \beta$ is the progressive (total) phase shift. Specify the condition for β for a

   i) broadside array; ii) end-fire-array; iii) phased (or scanning) array. (2p)

   c) A half-wavelength dipole has the input impedance $(73 + j42.5)\ \Omega$. What is the input impedance of a quarter-wavelength monopole placed directly above an infinite perfect electric conductor? (1p)

   d) A *folded* half-wavelength dipole has an input resistance of approximately

   i) 50 Ω; ii) 75 Ω; iii) 150 Ω; iv) 300 Ω; v) 600 Ω (1p)

Solution

a) $\mathcal{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{j\omega t}\}$

b) i) Broadside array: $\psi = kd\cos 90^\circ + \beta = 0 \quad \Rightarrow \quad \beta = 0$

ii) End-fire array: $\begin{cases} \psi = kd\cos 0^\circ + \beta = 0 & \Rightarrow \quad \beta = -kd \\ \psi = kd\cos 180^\circ + \beta = 0 & \Rightarrow \quad \beta = kd \end{cases}$

iii) Phased (or scanning) array: $\psi = kd\cos\theta_0 + \beta = 0 \quad \Rightarrow \quad \beta = -kd\cos\theta_0$

c) $Z_{\text{monopole}} = \frac{1}{2}Z_{\text{dipole}} = (36.5 + j21.25)\ \Omega$

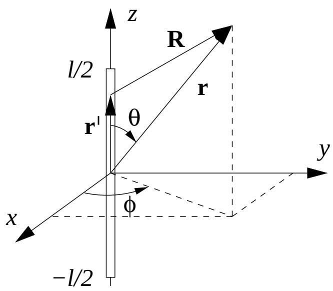
d) Impedance of a folded half-wavelength dipole: iv) 300Ω

2. Consider a very thin finite length dipole of length l which is symmetrically positioned about the origin with its length directed along the z axis according to the figure. In the far-field region the condition that the maximum phase error should be less than $\pi/8$ defines the inner boundary of that region to be $r = 2l^2/\lambda$. For $r \leq 2l^2/\lambda$, we are in the radiating near-field region and the far-field approximation is not valid. By allowing a maximum phase error of less than $\pi/8$, show that the inner boundary of this region is at $r = 0.62\sqrt{l^3/\lambda}$.

   Hint: The vector potential is given by

$$\mathbf{A} = \frac{\mu}{4\pi}\int \mathbf{I}_e(x',y',z')\frac{e^{-jkR}}{R}\mathrm{d}l' \, .$$

   Expand R, where the higher order terms become more important as the distance to the antenna decreases. Note that $\mathbf{r}' = z'\hat{z}$.



Solution

Apply the theorem of cosine to the triangle in the figure:

$$R^2 = r^2 + z'^2 - 2rz'\cos\theta \quad \text{or} \quad R = r\sqrt{1 - 2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2} \tag{1}$$

Expand the square root using

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \mathcal{O}(x^4) \quad \text{where} \quad x = -2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2 \tag{2}$$

$$\begin{aligned} R = r\Bigg\{ & 1 + \frac{1}{2}\left[-2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right] - \frac{1}{8}\left[-2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]^2 \\ & + \frac{1}{16}\left[-2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]^3 + \mathcal{O}\left[\left(\frac{z'}{r}\right)^4\right]\Bigg\} \end{aligned} \tag{3}$$

$$\begin{aligned} = r\Bigg\{ & 1 - \frac{z'}{r}\cos\theta + \frac{1}{2}\left(\frac{z'}{r}\right)^2 - \frac{1}{8}\left[\left(\frac{z'}{r}\right)^2\cos^2\theta - 4\left(\frac{z'}{r}\right)^3\cos\theta + \left(\frac{z'}{r}\right)^4\right] \\ & - \frac{1}{16}8\left(\frac{z'}{r}\right)^3\cos^3\theta + \mathcal{O}\left[\left(\frac{z'}{r}\right)^4\right]\Bigg\} \end{aligned} \tag{4}$$

$$= r\left\{1 - \frac{z'}{r}\cos\theta + \frac{1}{2}\left(\frac{z'}{r}\right)^2(1-\cos^2\theta) + \frac{1}{2}\left(\frac{z'}{r}\right)^3\cos\theta(1-\cos^2\theta) + \mathcal{O}\left[\left(\frac{z'}{r}\right)^4\right]\right\} \tag{5}$$

$$= r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right) + \dots \tag{6}$$

In the far-field region, the two first terms are used as the approximation for R and the third term is the error. In this case, we consider the radiating near-field region and we have to approximate R with the three first terms

$$R \approx r - z' \cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) \tag{7}$$

and the error is given by the fourth term

$$\frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right) \tag{8}$$

The maximum error is found when

$$\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\right] = \frac{z'^3}{2r^2}\sin\theta(-\sin^2\theta + 2\cos^2\theta) = 0 \tag{9}$$

We note that $\theta = 0$ or 180° give no maximum because they make the fourth term equal to zero. So, we must have the maximum error for

$$\theta = \arctan(\pm\sqrt{2}) \tag{10}$$

The maximum phase error $\leq \frac{\pi}{8}$, which gives

$$k\frac{1}{r^2}\frac{z'^3}{2}\cos\theta\sin^2\theta\Big|_{z'=l/2,\ \theta=\arctan(\pm\sqrt{2})} = \frac{2\pi}{\lambda}\frac{1}{r^2}\frac{1}{\sqrt{3}}\frac{2}{3} = \frac{\pi l^3}{12\sqrt{3}\lambda r^2} \leq \frac{\pi}{8} \tag{11}$$

We solve for r and obtain

$$r \geq 0.62\sqrt{\frac{l^3}{\lambda}} \tag{12}$$

which shows that the inner boundary of the radiating near-field region is

$$r = 0.62\sqrt{\frac{l^3}{\lambda}} \tag{13}$$

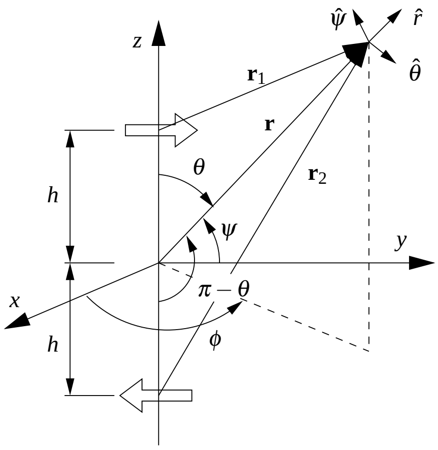
QED ■

3. An infinitesimal horizontal electric dipole of length l and constant electric current I_0 is placed parallel to the y axis a height $h = \lambda/2$ above an infinite electric ground plane.

   a) Find the spherical **E-** and **H-**field components radiated by the dipole in the far-zone.

   b) Find the angles of all the nulls of the total field.

Solution



a) Introduce a new set of spherical coordinates (r, ψ, χ), where $\hat{\psi}$ is given in the figure and $\hat{\chi} = \hat{r} \times \hat{\psi}$.

$$E^1_\psi \hat{\psi} = j\eta \frac{kI_0\, l}{4\pi r_1} e^{-jkr_1} \sin\psi\, \hat{\psi} \tag{1}$$

$$E^2_\psi \hat{\psi} = j\eta \frac{kI_0\, l}{4\pi r_2} e^{-jkr_2} \sin\psi\, \hat{\psi} \tag{2}$$

$$\begin{aligned} \sin\psi &= \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{r}\cdot\hat{y}|} \\ &= \sqrt{1-\sin^2\theta\sin^2\phi} \end{aligned} \tag{3}$$

Theorem of cosine:

$$\begin{array}{ll} r_1^2 = r^2+h^2-2rh\cos\theta & r_2^2 = r^2+h^2-2rh\cos(\pi-\theta) \\ r_1 = r\sqrt{1-2\frac{h}{r}\cos\theta+\left(\frac{h}{r}\right)^2} & r_2 = r\sqrt{1+2\frac{h}{r}\cos\theta+\left(\frac{h}{r}\right)^2}, \quad \text{but } \sqrt{1+x}\approx\frac{x}{2} \\ r_1 = r\left[1-\frac{h}{r}\cos\theta+\frac{1}{2}\left(\frac{h}{r}\right)^2\right] & r_2 = r\left[1+\frac{h}{r}\cos\theta+\frac{1}{2}\left(\frac{h}{r}\right)^2\right], \quad \text{but } \left(\frac{h}{r}\right)^2 \ll \frac{h}{r} \\ r_1 \approx r-h\cos\theta & r_2 \approx r+h\cos\theta \end{array} \tag{4}$$

Far-field approximation:

$$\left.\begin{array}{l} r_1 \approx r-h\cos\theta \\ r_1 \approx r+h\cos\theta \end{array}\right\} \quad \text{for phases} \tag{5}$$

$$r_1 \approx r_2 \approx r \qquad \text{for amplitudes} \tag{6}$$

$$\begin{aligned} \mathbf{E} = E^1_\psi\hat{\psi}+E^2_\psi\hat{\psi} &= j\eta\frac{kI_0\, l}{4\pi r}e^{-jkr}\sqrt{1-\sin^2\theta\sin^2\phi}\left[e^{jkh\cos\theta}-e^{-jkh\cos\theta}\right]\hat{\psi} = [kh=\pi] \\ &= \underbrace{j\eta\frac{kI_0\, l}{4\pi r}e^{-jkr}\sqrt{1-\sin^2\theta\sin^2\phi}}_{EF}\underbrace{[2j\sin(\pi\cos\theta)]}_{AF}\hat{\psi}, \qquad 0\le\theta\le\pi/2 \end{aligned} \tag{7}$$

$$\mathbf{H} = \frac{1}{\eta}\hat{r}\times\mathbf{E} = j\frac{kI_0\, l}{4\pi r}e^{-jkr}\sqrt{1-\sin^2\theta\sin^2\phi}\,[2j\sin(\pi\cos\theta)]\,\hat{\chi}, \qquad 0\le\theta\le\pi/2 \tag{8}$$

b) Nulls of the AF:

$$AF = 2j\sin(\pi\cos\theta) = 0 \tag{9}$$

$$\Rightarrow \quad \pi\cos\theta = n\pi, \qquad n = 0, \pm1, \pm2, \ldots \tag{10}$$

$$\Rightarrow \quad \cos\theta = n, \qquad n = -1, 0, 1 \tag{11}$$

$$
\begin{array}{lllll}
n=-1: & \cos\theta=-1 & \Rightarrow & \theta=180^\circ>90^\circ & \Rightarrow \quad \text{not a null} \\
n=0: & \cos\theta=0 & \Rightarrow & \theta=90^\circ & \\
n=1: & \cos\theta=1 & \Rightarrow & \theta=0^\circ &
\end{array}
\tag{12}
$$

Nulls of the EF:

$$1-\sin^2\theta\sin^2\phi=0 \tag{13}$$

$$\Rightarrow \quad \sin^2\theta\sin^2\phi=1 \tag{14}$$

$$\Rightarrow \quad \sin\theta=\pm1 \qquad \text{simultaneous with} \qquad \sin\phi=\pm1 \tag{15}$$

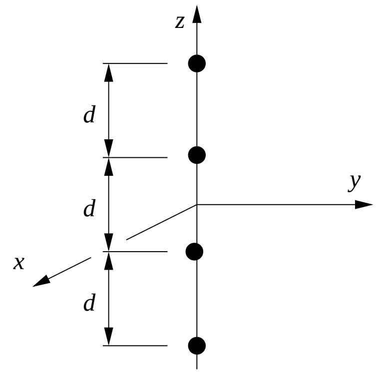
$$\Rightarrow \quad \theta=90^\circ \qquad \text{simultaneous with} \qquad \phi=90^\circ \tag{16}$$

$\theta=90^\circ$ is already a null for the AF, so the EF does not introduce any new nulls.

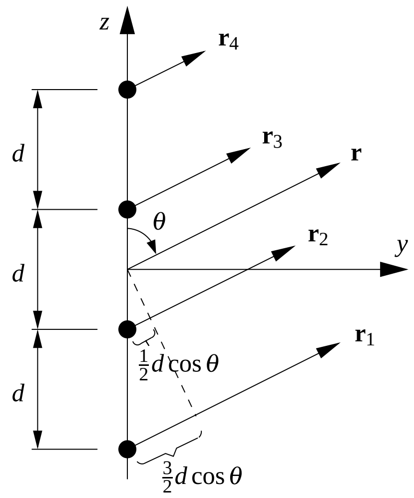
Nulls of the total field for $\theta=0^\circ$ and 90°, regardsless of value of ϕ.

4. A four-element uniform array has its elements placed along the z axis with distance $d = \lambda/2$ between them according to the figure below.

   a) Derive the array factor and show that it can be written as $\dfrac{\sin(2\psi)}{\sin(\psi/2)}$, where ψ is the progressive phase shift between the elements.
   b) In order to obtain maximum radiation along the direction $\theta = 0°$, where θ is measured from the positive z axis, determine the progressive phase shift ψ.
   c) Find all the nulls of the array factor.



Solution



a) $$E \sim \sum_{n=1}^{4} \frac{e^{-j(kr_n+\beta_n)}}{r_n} \approx \frac{e^{-jkr}}{r} AF \tag{1}$$

Far-field approximation for phases:

$$r_1 = r + \frac{3}{2} d\cos\theta \tag{2}$$

$$r_2 = r + \frac{1}{2} d\cos\theta \tag{3}$$

$$r_3 = r - \frac{1}{2} d\cos\theta \tag{4}$$

$$r_4 = r - \frac{3}{2} d\cos\theta \tag{5}$$

and for amplitudes:

$$r_1 \approx r_2 \approx r_3 \approx r_4 \approx r \tag{6}$$

$$AF = e^{-j\frac{3}{2}(kd\cos\theta+\beta)} + e^{-j\frac{1}{2}(kd\cos\theta+\beta)} + e^{j\frac{1}{2}(kd\cos\theta+\beta)} + e^{j\frac{3}{2}(kd\cos\theta+\beta)} \tag{7}$$

$$= [\psi = kp\cos\theta + \beta] = e^{-j\frac{3}{2}\psi} + e^{-j\frac{1}{2}\psi} + e^{j\frac{1}{2}\psi} + e^{j\frac{3}{2}\psi} \tag{8}$$

$$= e^{-j\frac{3}{2}\psi}\left(1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi}\right) \tag{9}$$

Use the expression for a geometric series or form $AF\, e^{j\psi} - AF$, where $AF\, e^{j\psi} = e^{-j\frac{3}{2}\psi}\left(e^{j\psi} + e^{j2\psi} + e^{j3\psi} + e^{j4\psi}\right)$.

$$AF\, e^{j\psi} - AF = e^{-j\frac{3}{2}\psi}\left(e^{j4\psi} - 1\right) \tag{10}$$

or

$$AF\left(e^{j\psi} - 1\right) = e^{-j\frac{3}{2}\psi}\left(e^{j4\psi} - 1\right) \tag{11}$$

Solving for the *AF* gives

$$AF = e^{-j\frac{3}{2}\psi}\frac{e^{j4\psi}-1}{e^{j\psi}-1} = e^{-j\frac{3}{2}\psi}\frac{e^{j2\psi}}{e^{j\frac{1}{2}\psi}}\frac{(e^{j2\psi}-e^{-j2\psi})}{(e^{j\frac{1}{2}\psi}-e^{-j\frac{1}{2}\psi})} = e^{-j\frac{3}{2}\psi}e^{j\frac{3}{2}\psi}\frac{\sin\psi}{\sin\frac{1}{2}\psi} \tag{12}$$

$$= \frac{\sin\psi}{\sin\frac{1}{2}\psi} \qquad \text{Q.E.D.} \tag{13}$$

b) For maximum along $\theta = 0^\circ$ all sources must be in phase, *i.e.*, $\psi(\theta = 0^\circ) = 0$:

$$\psi = (kd\cos\theta + \beta)|_{\theta=0^\circ,\, d=\lambda/2} = 0 \tag{14}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}\frac{\lambda}{2}\cos 0^\circ + \beta = 0 \quad \Rightarrow \quad \beta = -\pi \tag{15}$$

$$\Rightarrow \quad \psi = \pi(\cos\theta - 1) \tag{16}$$

c) Null of the *AF* $\quad \Rightarrow \quad AF = \frac{\sin 2\psi}{\sin\frac{\psi}{2}} = 0 \quad \Rightarrow \quad \sin 2\psi = 0$ **and** $\sin\frac{\psi}{2} \neq 0$
First we investigate the nominator:

$$\sin[2\pi(\cos\theta - 1)] = 0 \tag{17}$$

$$\Rightarrow \quad 2\pi(\cos\theta - 1) = n\pi, \qquad n = 0, \pm 1, \ldots \tag{18}$$

$$\Rightarrow \quad \cos\theta - 1 = \frac{n}{2} \tag{19}$$

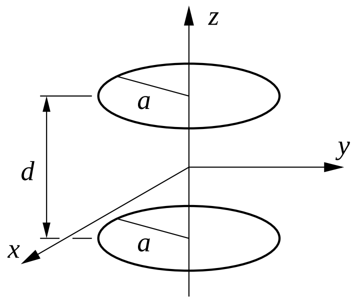
$$\Rightarrow \quad \cos\theta = 1 + \frac{n}{2} = \frac{2 = n}{2}, \qquad n = -4,\ -3,\ -2,\ -1,\ 0 \tag{20}$$

Then we insert the values of n and obtain the angle θ and simultaneously check the denominator:

$$\begin{array}{llllllllll}
n = -4 & \Rightarrow & \cos\theta = -1 & \Rightarrow & \theta = 180^\circ & \Rightarrow & \sin\left[\frac{\pi}{2}(\cos\theta - 1)\right] = 0 & \Rightarrow & \text{Not a null!} \\
n = -3 & \Rightarrow & \cos\theta = -1/2 & \Rightarrow & \theta = 120^\circ & \Rightarrow & \sin\left[\frac{\pi}{2}(\cos\theta - 1)\right] \neq 0 & & \text{OK!} \\
n = -2 & \Rightarrow & \cos\theta = 0 & \Rightarrow & \theta = 90^\circ & \Rightarrow & \sin\left[\frac{\pi}{2}(\cos\theta - 1)\right] \neq 0 & & \text{OK!} \\
n = -1 & \Rightarrow & \cos\theta = 1/2 & \Rightarrow & \theta = 60^\circ & \Rightarrow & \sin\left[\frac{\pi}{2}(\cos\theta - 1)\right] \neq 0 & & \text{OK!} \\
n = -0 & \Rightarrow & \cos\theta = 1 & \Rightarrow & \theta = 0^\circ & \Rightarrow & \sin\left[\frac{\pi}{2}(\cos\theta - 1)\right] = 0 & \Rightarrow & \text{Not a null!}
\end{array} \tag{21}$$

Nulls for $\theta = 60^\circ,\ 90^\circ,\ 120^\circ$

5. Two identical constant current loops with radius a are placed a distance d apart according to the figure below. Determine the smallest radius a and the smallest separation d so that nulls are formed in the directions $\theta = 0^\circ$, 60°, 90°, 120°, and 180°, where θ is the angle measured from the positive z axis.



Solution

Study the element factor(EF) and the array factor (AF) separately.

EF: The electric field from a constant current loop is

$$E_\phi = \eta \frac{kaI_0}{2r} e^{-jkr} J_1(ka \sin\theta) \tag{1}$$

The Bessel function $J_1(ka\sin\theta)$ has zeros for $ka\sin\theta = 0,\ 3.8317, \ldots$

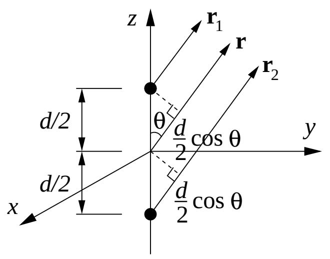
$$ka\sin\theta = 0 \qquad \Rightarrow \qquad \theta = 0^\circ \quad \text{or} \quad \theta = 180^\circ \tag{2}$$

$$ka\sin\theta = 3.8317 \quad \Rightarrow \quad a = \frac{3.8317}{k\sin\theta} = \frac{3.8317\lambda}{2\pi\sin\theta} \tag{3}$$

Since the elements are fed with equal phases (and amplitudes), the AF must have a maximum in the broadside direction $\theta = 90^\circ$ and not a null. Therefore, the null for $\theta = 90^\circ$ must come from the EF, so

$$a = \frac{3.8317}{2\pi\sin 90^\circ} = 0.61\lambda \tag{4}$$

AF We must choose d so that $AF(\theta = 60^\circ = AF(\theta = 120^\circ) = 0$



Far-field approximation:

$$\left.\begin{array}{l} r_1 = r - \frac{d}{2}\cos\theta \\ r_2 = r + \frac{d}{2}\cos\theta \end{array}\right\} \quad \text{for phases} \tag{5}$$

$$r_1 \approx r_2 \approx r \qquad \text{for amplitudes} \tag{6}$$

$$AF = e^{jk\frac{d}{2}\cos\theta} + e^{-jk\frac{d}{2}\cos\theta} = 2\cos\left(\frac{kd}{2}\cos\theta\right) \tag{7}$$

$$AF(\theta = 60^\circ) = 0 \quad \Rightarrow \quad \cos\left(\frac{\pi d}{\lambda}\cos 60^\circ\right) = \cos\left(\frac{\pi d}{2\lambda}\right) = 0 \tag{8}$$

$$\Rightarrow \quad \frac{\pi d}{2\lambda} = (2n+1)\frac{\pi}{2}, \qquad n = 0, \pm1, \pm2, \ldots \tag{9}$$

$$\Rightarrow \quad d = (2n+1)\lambda, \qquad n = 0, \pm1, \pm2, \ldots \tag{10}$$

Choose $d = \lambda$ as the smallest distance.

$$AF = 2\cos\left(\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta\right) = 2\cos(\pi\cos\theta) \tag{11}$$

We have to check the zero at $\theta = 120^\circ$:

$$AF(\theta = 120^\circ) = 2\cos(\pi\cos 120^\circ) = 2\cos\left(-\frac{\pi}{2}\right) = 0 \qquad \text{OK!} \tag{12}$$

For zeros along $\theta = 0^\circ$, $\theta = 60^\circ$, $\theta = 90^\circ$, $\theta = 120^\circ$, and $\theta = 180^\circ$, we choose the radius of the loops to be $a = 0.61\lambda$ and the spacing between the loops as $d = \lambda$.

6. Design a linear array of isotropic elements placed along the z axis such that the nulls of the array factor occur at $\theta = 60^\circ$, $\theta = 90^\circ$, and $\theta = 120^\circ$. Assume that the elements are spaced a distance $d = \lambda/4$ apart and that $\beta = 45^\circ$.

   a) Sketch and label the visible region on the unit circle.
   b) Find the required number of elements.
   c) Determine the excitation coefficients.

   Hint: The array factor of an N-element linear array is given by $AF = \sum_{n=1}^{N} a_n e^{\mathrm{j}(n-1)\psi}$, where $\psi = kd\cos\theta + \beta$. Use the representation $z = e^{\mathrm{j}\psi}$.

Solution

a) The visible region:

$$AF = \sum_{n=1}^{N} a_n e^{j(n-1)\psi} = \sum_{n=1}^{N} a_n z^{n-1} = a_1 + a_2 z + \cdots + a_N z^{N-1}$$
$$= (z - z_1)(z - z_2)\cdot\cdots\cdot(z - z_{N-1}) \tag{1}$$

$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\frac{5\lambda}{8}\cos\theta - \frac{\pi}{4} = \frac{\pi}{4}(5\cos\theta - 1) \tag{2}$$

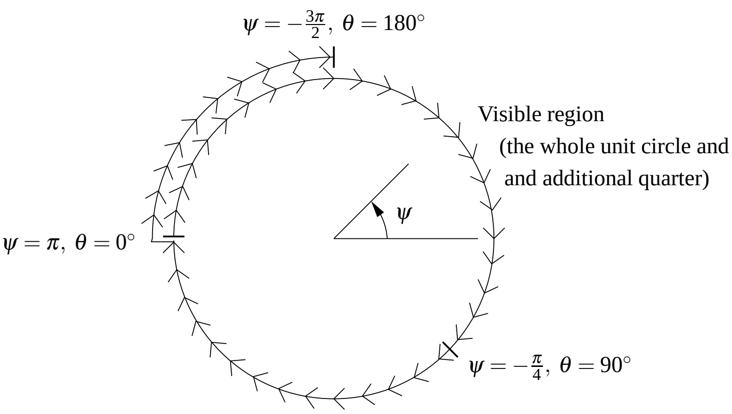
$$\theta = 0^\circ \quad \Rightarrow \quad \psi = \frac{\pi}{4}(5-1) = \pi \tag{3}$$

$$\theta = 90^\circ \quad \Rightarrow \quad \psi = \frac{\pi}{4}(0-1) = -\frac{\pi}{4} \tag{4}$$

$$\theta = 180^\circ \quad \Rightarrow \quad \psi = \frac{\pi}{4}(-5-1) = -\frac{3\pi}{2} \tag{5}$$

$$z = e^{j\psi} \quad \Rightarrow \quad |z| = 1 \quad \Rightarrow \quad \text{unit circle} \tag{6}$$

When θ varies from 0° to 180°, ψ varies from π to $-3\pi/2$



b) Nulls:

$$\theta_1 = 0^\circ \quad \Rightarrow \quad \psi_1 = \pi \quad \Rightarrow \quad z_1 = e^{j\psi_1} = e^{j\pi} = -1 \tag{7}$$

$$\theta_2 = 90^\circ \quad \Rightarrow \quad \psi_2 = -\frac{\pi}{4} \quad \Rightarrow \quad z_2 = e^{j\psi_2} = e^{-j\pi/4} = \frac{1}{\sqrt{2}}(1-j) \tag{8}$$

$$\theta_3 = 180^\circ \quad \Rightarrow \quad \psi_3 = -\frac{3\pi}{2} \quad \Rightarrow \quad z_3 = e^{j\psi_3} = e^{-j\frac{3\pi}{2}} = j \tag{9}$$

$$AF = (z - z_1)(z - z_2)(z - z_3) = a_1 + a_2 z + a_3 z^2 + a_4 z^3 \tag{10}$$

$$\Rightarrow \quad \text{4 elements } (a_1, a_2, a_3, a_4) \text{ are needed} \tag{11}$$

c) Excitation coefficients:

$$\begin{aligned}(z - z_1)(z - z_2)(z - z_3) &= \dots \\ &= z^3 - z^2 \underbrace{(z_1 + z_2 + z_3)}_{-a_3} + z \underbrace{(z_1 z_2 + z_2 z_3 + z_3 z_1)}_{a_2} \underbrace{- z_1 z_2 z_3}_{a_1}\end{aligned} \tag{12}$$

$$a_1 = -z_1 z_2 z_3 = -(-1)\frac{1}{\sqrt{2}}(1 - j)j = \frac{1}{\sqrt{2}}(1 + j) \tag{13}$$

$$a_2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = (-1)\frac{1}{\sqrt{2}}(1 - j) + \frac{1}{\sqrt{2}}(1 - j)j + j(-1) = j\left(\sqrt{2} - 1\right) \tag{14}$$

$$a_3 = -(z_1 + z_2 + z_3) = -\left(-1 + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + j\right) = \left(1 - \frac{1}{\sqrt{2}}\right)(1 - j) \tag{15}$$

$$a_4 = 1 \tag{16}$$