Solution to Exam in *Antenna Theory*
Time: 17 March 2009, at 8.00–13.00.
Location: Gimogatan 4, Sal 2

You may bring: Laboratory reports, pocket calculator, English dictionary, Råde-Westergren: “Beta”, Nordling-Österman: “Physics Handbook”, or comparable handbooks.

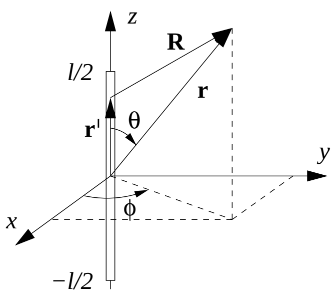
Six problems, maximum five points each, for a total maximum of 30 points.

1. Consider an infinitesimal electric dipole of length l placed symmetically at the origin. Assume that the current in the dipole is constant and given by $\mathbf{I}_\mathrm{e}(z') = \hat{\mathbf{z}} I_0$. Find the electric and magnetic field components radiated by the dipole in *all* space.
   Hint: The vector potential $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{\mu}{4\pi} \int \mathbf{I}_\mathrm{e}(x', y', z') \frac{e^{-\mathrm{j}kR}}{R} \mathrm{d}l'$$

where R is the distance from any point on the source to the observation point.

Solution



Infinitesimal dipole:

$$\mathbf{I}_e = \hat{z} I_0 \quad (1)$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' \quad (2)$$

$$\mathbf{r}' = (x', y', z') = \mathbf{0} \quad \text{(for infinitesimal dipole)} \quad (3)$$

$$\mathbf{R} = \mathbf{r} \quad \text{for phase and amplitude} \quad (4)$$

The vector potential becomes

$$\mathbf{A} = \hat{z} \frac{\mu I_0 e^{-jkr}}{4\pi r} \int_{-l/2}^{l/2} \mathrm{d}z' = \hat{z} \frac{\mu I_0 l}{4\pi r} e^{-jkr} \quad (5)$$

Change to spherical coordinates, $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$$\mathbf{A} = \frac{\mu I_0 l}{4\pi r} e^{-jkr} (\hat{r}\cos\theta - \hat{\theta}\sin\theta) \quad (6)$$

The magnetic field

$$\begin{aligned}
\mathbf{H} &= \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{\mu} \left\{ \hat{r} \cdot 0 + \hat{\theta} \cdot 0 + \hat{\phi} \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \right\} \\
&= \hat{\phi} \frac{1}{\mu r} \left[\frac{\partial}{\partial r} \left(-\frac{\mu I_0 l}{4\pi} e^{-jkr} \sin\theta \right) - \frac{\partial}{\partial \theta} \left(\frac{\mu I_0 l}{4\pi r} e^{-jkr} \cos\theta \right) \right] \\
&= \hat{\phi} \frac{1}{\mu r} \left[\frac{\mu I_0 l}{4\pi} e^{-jkr} jk \sin\theta + \frac{\mu I_0 l}{4\pi r} e^{-jkr} \sin\theta \right] \\
&= \hat{\phi} \frac{I_0 l}{4\pi r} e^{-jkr} \left[jk \sin\theta + jk \frac{\sin\theta}{jkr} \right] = \hat{\phi}\, j \frac{k I_0 l}{4\pi r} \sin\theta \left[1 + \frac{1}{jkr} \right] e^{-jkr}
\end{aligned} \quad (7)$$

The electric field

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} \qquad\qquad \left[\mathbf{H}=(0,0,H_\phi)\right] \\
&= \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta)\right]+\frac{\hat{\theta}}{r}\left[-\frac{\partial}{\partial r}(rH_\phi)\right]\right\} \\
&= \frac{1}{j\omega\varepsilon}j\frac{kI_0l}{4\pi}\left\{\frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin^2\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]+\frac{\hat{\theta}}{r}\left[-\frac{\partial}{\partial r}\left(\sin\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]\right\} \\
&= \frac{1}{\omega\varepsilon}\frac{kI_0l}{4\pi}\left\{\frac{\hat{r}}{r^2\sin\theta}\left[1+\frac{1}{jkr}\right]e^{-jkr}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)-\frac{\hat{\theta}}{r}\sin\theta\frac{\partial}{\partial r}\left(\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right\} \\
&= \underbrace{\frac{k}{\omega\varepsilon}}_{\eta}\frac{I_0l}{4\pi}\left\{\frac{\hat{r}}{r^2\sin\theta}\left[1+\frac{1}{jkr}\right]e^{-jkr}2\sin\theta\cos\theta-\frac{\hat{\theta}}{r}\sin\theta\left[-jk-\frac{1}{r}-\frac{1}{jkr^2}\right]e^{-jkr}\right\} \\
&= \eta\frac{I_0l}{4\pi}\left\{\hat{r}\frac{2}{r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}\cos\theta+\hat{\theta}\frac{jk}{r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}\sin\theta\right\} \\
&= \hat{r}\,\eta\frac{I_0l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}+\hat{\theta}\,j\eta\frac{kI_0l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr} \qquad (8)
\end{aligned}
$$

We have found the fields valid in all space from an infinitesimal dipole:

$$H_\phi = j\frac{kI_0l}{4\pi r}\sin\theta\left[1+\frac{1}{jkr}\right]e^{-jkr} \qquad (9)$$

$$H_r = H_\theta = E_\phi = 0 \qquad (10)$$

$$E_r = \eta\frac{I_0l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr} \qquad (11)$$

$$E_\theta = j\eta\frac{kI_0l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr} \qquad (12)$$

2. The normalized far-zone field pattern of an antenna is given by

$$E = \begin{cases} \sqrt{\sin\theta \sin^2\phi} & 0 \le \theta \le \pi,\ \ 0 \le \phi \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

Find the directivity using

a) the exact expression;

b) the most appropriate approximative formula.

c) Explain why your chosen approximative formula gives the best value. When should the other approximative formula be prefered?

Solution

a) Normalised radiation intensity $U = \sin\theta \sin^2\phi \qquad \begin{matrix} 0 \le \theta \le \pi \\ 0 \le \phi \le \pi \end{matrix}$

This is a directive radiation pattern.

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} \tag{1}$$

$$P_{\text{rad}} = \int U \,\mathrm{d}\Omega = \int_0^\pi \int_0^\pi \sin^2\phi \sin^2\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = \int_0^\pi \sin^2\phi \,\mathrm{d}\phi \int_0^\pi \sin^2\theta \,\mathrm{d}\theta = \frac{\pi^2}{4} \tag{2}$$

$$U_{\max} = 1 \tag{3}$$

$$D_0 = \frac{4\pi}{\pi^2/4} = \frac{16}{\pi} \approx 5.09 \tag{4}$$

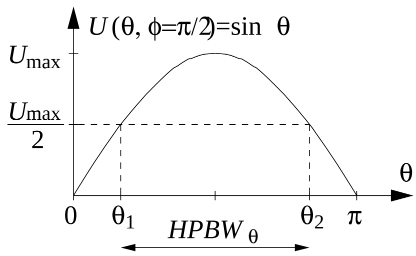
b) The pattern is not very directive. Therefore, Kraus' approximative method is more accurate than Tai-Pereira's method. We use Kraus' method:

$$D_0 = \frac{4\pi}{\Theta_{1r}\Theta_{2r}} \qquad \begin{matrix} \Theta_{1r} = HPBW_\theta \\ \Theta_{2r} = HPBW_\phi \end{matrix} \tag{5}$$

Find $HPBW_\theta$ by setting $\phi = \phi_{\max} = \pi/2$ and solving for half the max. radiation intensity:

$$\frac{U_{\max}}{2} = \frac{1}{2} = \sin\theta \quad \Rightarrow \quad \begin{matrix} \theta_1 = 30^\circ \\ \theta_2 = 150^\circ \end{matrix}$$

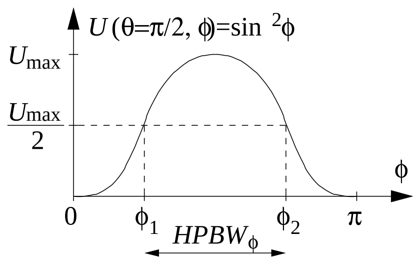
$$HPBW_\theta = \theta_2 - \theta_1 = 120^\circ = \frac{2\pi}{3} \quad \text{(elevation plane)}$$



Find $HPBW_\phi$ by setting $\theta = \theta_{\max} = \pi/2$ and solving for half the max. radiation intensity:

$$\frac{U_{\max}}{2} = \frac{1}{2} = \sin^2\phi \quad \Rightarrow \quad \begin{matrix} \phi_1 = 45^\circ \\ \phi_2 = 135^\circ \end{matrix}$$

$$HPBW_\phi = \phi_2 - \phi_1 = 90^\circ = \frac{\pi}{2} \quad \text{(azimuthal plane)}$$



Kraus: $\quad D_0 = \dfrac{4\pi}{(2\pi/3)(\pi/2)} = \dfrac{12}{\pi} \approx 3.82 \qquad$ (Tai-Pereira gives $D_0 = 3.24$)

c) Kraus is best for less directive patterns as the one in the problem, while Tai-Pereira is best for very directive patterns.

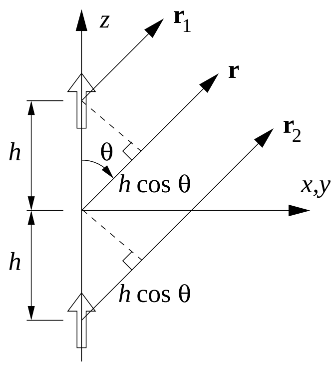
Comment: McDonald's and Pozar's formulae are for *omnidirectional* patterns and cannot be used.

3. A vertical infinitesimal linear electric dipole of length l is placed a distance h above an infinite perfectly conducting electric ground plane.

   a) Find the electric and magnetic fields radiated by the dipole in the presence of the ground plane.

   b) Determine the height h above the conducting plane at which the dipole must be elevated so that nulls are formed at angles $\theta = 0^\circ$ and $\theta = 60^\circ$ from the direction normal to the conducting plane.

Solution



a)
$$\mathbf{E}_1 = j\eta \frac{kI_0\, l}{4\pi r_1} e^{-jkr_1} \sin\theta\, \hat{\theta} \tag{1}$$

$$\mathbf{E}_2 = j\eta \frac{kI_0\, l}{4\pi r_2} e^{-jkr_2} \sin\theta\, \hat{\theta} \tag{2}$$

Far-field approximation:

$$\left.\begin{aligned} r_1 &\approx r - h\cos\theta \\ r_2 &\approx r + h\cos\theta \end{aligned}\right\} \quad \text{for phases} \tag{3}$$

$$r_1 \approx r_2 \approx r \qquad \text{for amplitudes} \tag{4}$$

$$\begin{aligned} \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 &= j\eta \frac{kI_0\, l}{4\pi r} e^{-jkr} \sin\theta \left(e^{jkh\cos\theta} + e^{-jkh\cos\theta} \right) \hat{\theta} \\ &= \underbrace{j\eta \frac{kI_0\, l}{4\pi r} e^{-jkr} \sin\theta}_{EF} \underbrace{\left[2\cos(kh\cos\theta)\right]}_{AF} \hat{\theta}, \qquad 0 \le \theta \le \pi/2 \end{aligned} \tag{5}$$

$$\mathbf{H} = \frac{1}{\eta} \hat{r} \times \mathbf{E} = j \frac{kI_0\, l}{4\pi r} e^{-jkr} \sin\theta \left[2\cos(kh\cos\theta)\right] \hat{\phi}, \qquad 0 \le \theta \le \pi/2 \tag{6}$$

b) Null at $\theta = 0^\circ$ from EF

Null at $\theta = 60^\circ$ if $AF(\theta = 60^\circ) = 0 \quad \Rightarrow$

$$\cos(kh\cos 60^\circ) = 0, \qquad kh = \frac{2\pi}{\lambda} h, \quad \cos 60^\circ = \frac{1}{2} \tag{7}$$

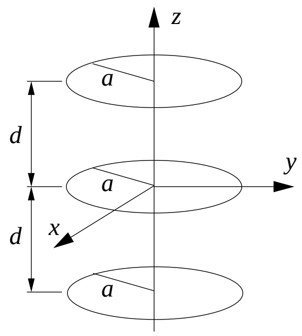
$$\cos\left(\frac{\pi h}{\lambda}\right) = 0 \tag{8}$$

$$\frac{\pi h}{\lambda} = \pm \frac{\pi}{2} + 2\pi n, \qquad n = 0, \pm 1, \ldots \tag{9}$$

$$h = \frac{\lambda}{2} + 2n\lambda, \qquad n = 0, \pm 1, \ldots \tag{10}$$

$$\text{Smallest height } h = \frac{\lambda}{2} \tag{11}$$

4. Three constant current circular loops of equal radii $a = 7\lambda/10$ and equal current amplitudes and phases are placed along the z axis with the planes ot the loops in the xy plane, see figure below. The spacing between the loops is uniform and equal to $d = \lambda/2$. If the loops are assumed not to couple to each other, find the nulls of the far-field pattern radiated by the loops.



Solution

Total field $= EF \times AF$ and the nulls of the array are the nulls of the element factor and the nulls of the array factor.

Element Factor:

$$E_\phi = \eta \frac{kaI_0}{2r} e^{-jkr} J_1(ka \sin\theta) \sim J_1(ka\sin\theta) \tag{1}$$

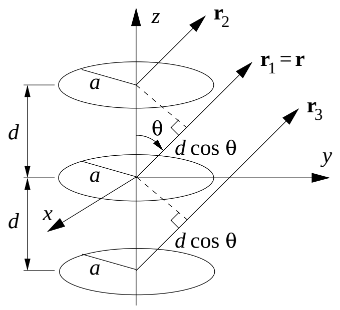
$J_1(x) = 0$ for $x_0 = 0,\ x_1 = 3.8317,\ x_2 = 7.0155,\ x_3 = 10.1743, \ldots$

$$x_0 = 0: \quad ka\sin\theta = \frac{7\pi}{5}\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \begin{cases} \theta_1 = 0^\circ \\ \theta_2 = 180^\circ \end{cases} \tag{2}$$

$$x_1 = 3.8317: \quad \frac{7\pi}{5}\sin\theta = 3.8317 \quad \Rightarrow \quad \sin\theta = 0.871 \quad \Rightarrow \quad \begin{cases} \theta_3 = 60.6^\circ \\ \theta_4 = 119.4^\circ \end{cases} \tag{3}$$

$$x_2 = 7.0155: \quad \frac{7\pi}{5}\sin\theta = 7.0155 \quad \Rightarrow \quad \sin\theta = 1.60 > 1 \quad \Rightarrow \quad \text{no more nulls} \tag{4}$$

Array Factor:



Far-field approximation:

$$\left.\begin{aligned} r_2 &\approx r - d\cos\theta \\ r_3 &\approx r + d\cos\theta \end{aligned}\right\} \quad \text{for phases} \tag{5}$$

$$r_2 \approx r_3 \approx r \qquad \text{for amplitudes} \tag{6}$$

$$AF = 1 + e^{jkd\cos\theta} + e^{-jkd\cos\theta} = 1 + 2\cos(kd\cos\theta) \tag{7}$$

$$AF = 0 \quad \Rightarrow \quad \cos(kd\cos\theta) = -1/2 \tag{8}$$

$$kd\cos\theta = \pm\frac{2\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \ldots \qquad kd = \pi \tag{9}$$

$$\cos\theta = \pm\frac{2}{3} + 2n, \quad n = 0 \tag{10}$$

$$\theta_5 = 48.2^\circ, \quad \theta_6 = 131.8^\circ \tag{11}$$

Nulls at $\theta = 0^\circ,\ 48.2^\circ,\ 60.6^\circ,\ 119.4^\circ,\ 131.8^\circ,\ 180^\circ$

5. Design a linear array of isotropic elements placed along the z axis such that the nulls of the array factor occur at $\theta = 0^\circ$, 60°, 90°, 120°, and 180°. Assume that the elements are spaced a distance $d = \lambda/2$ apart and that $\beta = 0$.

   a) Sketch and label the visible region on the unit circle.

   b) Find the smallest possible number of required elements and their excitation coefficients.

   c) Determine the length of the array.

   Hint: The array factor of an N-element linear array is given by $AF = \sum_{n=1}^{N} a_n e^{j(n-1)\psi}$, where $\psi = kd\cos\theta + \beta$. Use the representation $z = e^{j\psi}$.

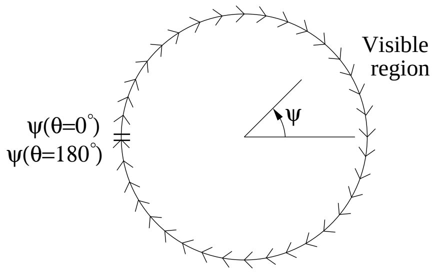
Solution

a) $$AF = \sum_{n=1}^{N} a_n e^{j(n-1)\psi} = \sum_{n=1}^{N} a_n z^{n-1} = a_1 + a_2 z + \cdots + a_N z^{N-1} = (z - z_1)\cdot \cdots \cdot (z - z_{N-1})$$

$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta = \pi\cos\theta$$

$$z = e^{j\psi} \quad \Rightarrow \quad |z| = 1 \quad \Rightarrow \quad \text{unit circle}$$

$$\begin{array}{lll} \theta = 0^\circ & \Rightarrow & \psi = \pi \\ \theta = 180^\circ & \Rightarrow & \psi = -\pi \\ \theta = 90^\circ & \Rightarrow & \psi = 0 \\ \theta = 60^\circ & \Rightarrow & \psi = \pi/2 \end{array}$$



b) Nulls:

$$\begin{array}{lllll} \theta_1 = 0^\circ & \Rightarrow & \psi_1 = \pi & \Rightarrow & z_1 = e^{j\pi} = -1 \\ \theta_2 = 60^\circ & \Rightarrow & \psi_2 = \pi/2 & \Rightarrow & z_2 = e^{j\pi/2} = j \\ \theta_3 = 90^\circ & \Rightarrow & \psi_3 = 0 & \Rightarrow & z_3 = e^{j0} = 1 \\ \theta_4 = 120^\circ & \Rightarrow & \psi_4 = -\pi/2 & \Rightarrow & z_4 = e^{-j\pi/2} = -j \\ \theta_5 = 180^\circ & \Rightarrow & \psi_5 = -\pi & \Rightarrow & z_5 = e^{-j\pi} = -1 = z_1 \end{array}$$

Because $z_5 = z_1$ we omit z_5 for the smallest number of elements

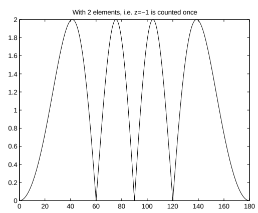
$$AF = (z+1)(z-1)(z-j)(z+j) = (z^2-1)(z^2+1) = z^4 - 1$$

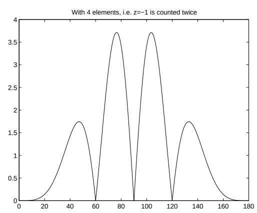
Identification of coefficients give only 2 required elements:

$$\begin{array}{l} a_1 = -1 \\ a_2 = a^3 = a^4 = 0 \\ a_5 = 1 \end{array}$$

c) The length of the array is the distance between elements #1 and #5, *i.e.*, $4d = 2\lambda$

Comment: If we take into account z_5, *i.e.* we count $z = -1$ twice, we obtain 4 elements. This would give the same number and location of the nulls, but a different radiation pattern, see the figures below. The first figure is for 2 elements and the second for 4 elements.





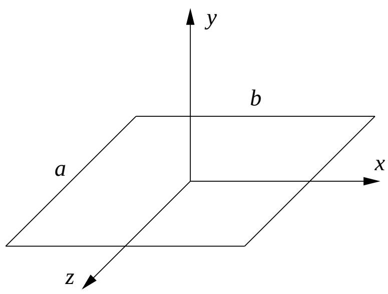
6. A rectangular aperture of dimensions a and b is placed at an infinite ground plane as shown in the figure below. The tangential field distribution over the aperture is given by

$$\mathbf{E}_a = \hat{x}E_0 \qquad \begin{matrix} -b/2 \le x' \le b/2 \\ -a/2 \le z' \le a/2 \end{matrix}$$

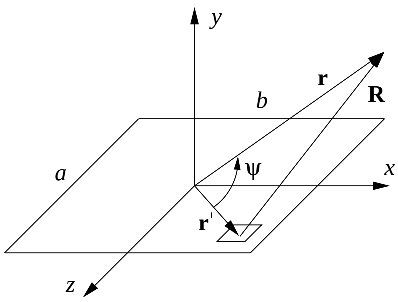
Find the spherical far-zone electric and magnetic field components radiated by the aperture. The spherical field components must be expressed with respect to the coordinate system specified in the figure.

Hint:

$$\int_{-c/2}^{c/2} e^{j\alpha x}\mathrm{d}x = c\,\frac{\sin\left(\frac{\alpha}{2}c\right)}{\frac{\alpha}{2}c}$$



Solution



$$\mathbf{E}_a = \hat{x}E_0 \qquad \begin{cases} -b/2 \le x' \le b/2 \\ -a/2 \le z' \le a/2 \end{cases} \tag{1}$$

$$\text{Far-zone fields:} \quad \begin{cases} \mathbf{E} = (0, E_\theta, E_\phi), \;\; E_r = 0 \\ \mathbf{H} = (0, H_\theta, H_\phi), \;\; H_r = 0 \end{cases} \tag{2}$$

The equivalence principle gives the electric conductor equivalent (infinite electric ground plane), see the formula collection.

The equivalent current densities are

$$\mathbf{M}_s = \begin{cases} -2\hat{n}\times\mathbf{E}_a = -2\hat{y}\times\hat{x}E_0 = \hat{z}\,2E_0 \quad \text{over the aperture} \\ \mathbf{0} \quad \text{elsewhere} \end{cases} \tag{3}$$

$$\mathbf{J}_s = \mathbf{0} \quad \text{everywhere} \tag{4}$$

Theorem of cosine:

$$R^2 = r^2 + r'^2 - 2rr'\cos\psi \tag{5}$$

$$R = r\sqrt{1 - 2\frac{r'}{r}\cos\psi + \left(\frac{r'}{r}\right)^2} = \left[\left(\frac{r'}{r}\right)^2 \ll \frac{r'}{r},\;\; \sqrt{1+x} \approx 1 + \frac{x}{2} \;\text{ when }\; x \ll 1\right]$$

$$\approx r\left(1 - \frac{r'}{r}\cos\psi\right) = r - r'\cos\psi \tag{6}$$

But $r'\cos\psi = \mathbf{r}'\cdot\hat{r} = (x'\hat{x}+z'\hat{z})\cdot(\hat{x}\sin\theta\cos\phi+\hat{y}\sin\theta\sin\phi+\hat{z}\cos\theta) = x'\sin\theta\cos\phi+z'\cos\theta$

Far-field approximation: $R \approx r - x'\sin\theta\cos\phi - z'\cos\theta$ for phases
. $R \approx r$ for amplitudes

$$\mathbf{F} \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\mathbf{L} \tag{7}$$

$$\mathbf{L} = \int_{S'} \mathbf{M}_s e^{jk(x'\sin\theta\cos\phi+z'\cos\theta)}\mathrm{d}x'\mathrm{d}z' \tag{8}$$

$$\mathbf{M}_s = \hat{z}2E_0 = 2E_0(\hat{r}\cos\theta - \hat{\theta}\sin\theta) \quad \Rightarrow \quad \mathbf{L} = (L_r, L_\theta, 0) \tag{9}$$

Radiated fields:

$$\mathbf{H} = -j\omega\mathbf{F} \quad \text{for } \theta \text{ and } \phi,\ H_r = 0 \tag{10}$$

$$\left[\mathbf{H} = \frac{1}{\eta}\hat{r}\times\mathbf{E} \quad \Rightarrow \quad \hat{r}\times\mathbf{H} = \frac{1}{\eta}\hat{r}\times(\hat{r}\times\mathbf{E}) = \frac{1}{\eta}[\hat{r}\underbrace{(\hat{r}\cdot\mathbf{E})}_{0} - \mathbf{E}\underbrace{(\hat{r}\cdot\hat{r})}_{1}] = -\frac{1}{\eta}\mathbf{E}\right] \tag{11}$$

$$\mathbf{E} = -\eta\hat{r}\times\mathbf{H} = j\omega\eta(\hat{r}\times\mathbf{F}) \quad \text{for } \theta \text{ and } \phi,\ E_r = 0 \tag{12}$$

We do not need to calculate L_r. In addition, $L_\phi = 0$, so

$$L_\theta = -2E_0\sin\theta\int_{-b/2}^{b/2} e^{jkx'\sin\theta\cos\phi}\,\mathrm{d}x'\int_{-a/2}^{a/2} e^{jkz'\cos\theta}\,\mathrm{d}z' \tag{13}$$

Use the hint: $\displaystyle\int_{-c/2}^{c/2} e^{j\alpha x}\,\mathrm{d}x = c\,\frac{\sin\left(\frac{\alpha}{2}c\right)}{\frac{\alpha}{2}c}$

$$L_\theta = -2E_0\sin\theta\, b\frac{\sin\left(\frac{kb}{2}\sin\theta\cos\phi\right)}{\frac{kb}{2}\sin\theta\cos\phi}\, a\,\frac{\sin\left(\frac{ka}{2}\cos\theta\right)}{\frac{ka}{2}\cos\theta} = -2abE_0\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z} \tag{14}$$

where

$$X = \frac{kb}{2}\sin\theta\cos\phi \qquad \text{and} \qquad Z = \frac{ka}{2}\cos\theta \tag{15}$$

$$\mathbf{F} \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}L_\theta\hat{\theta} = -\hat{\theta}\,\frac{ab\varepsilon E_0}{2\pi r}e^{-jkr}\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z} \tag{16}$$

The fields:

$$H_\theta \approx -j\omega F_\theta \approx j\omega\frac{ab\varepsilon E_0}{2\pi r}e^{-jkr}\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z} = \left[\omega\varepsilon = kc\varepsilon = k\frac{\varepsilon}{\sqrt{\varepsilon\mu}} = \frac{k}{\eta}\right]$$

$$= j\frac{abkE_0}{2\pi\eta r}e^{-jkr}\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z} \tag{17}$$

$$H_\phi = -j\omega F_\phi = 0 \tag{18}$$

$$\mathbf{E} = j\omega\eta(\hat{r}\times\mathbf{F}) = -\eta\hat{r}\times\mathbf{H} \tag{19}$$

$$E_\theta = 0 \tag{20}$$

$$E_\phi = -j\frac{abkE_0}{2\pi r}e^{-jkr}\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z} \tag{21}$$
