Exam in *Antenna Theory*
*Time:* 31 March 2008, at 8.00–13.00.
*Location:* Polacksbacken, Skrivsal

*You may bring*: Laboratory reports, pocket calculator, English dictionary, Råde-Westergren: "Beta", Nordling-Österman: "Physics Handbook", or comparable handbooks.

Six problems, maximum five points each, for a total maximum of 30 points.

1. Consider an infinitesimal electric dipole of length $l$ placed symmetically at the origin. Assume that the current in the dipole is constant and given by $\mathbf{I}_e(z') = \hat{\mathbf{z}}I_0$. Find the electric and magnetic field components radiated by the dipole in *all* space.
Hints: The vector potential $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{\mu}{4\pi}\int \mathbf{I}_e(x',y',z')\frac{e^{-jkR}}{R}dl'$$

where $R$ is the distance from any point on the source to the observation point. The magnetic and electric fields are given by

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu}\nabla\times\mathbf{A}\\ \mathbf{E} &= -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\mathbf{A}) = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H}\end{aligned}$$

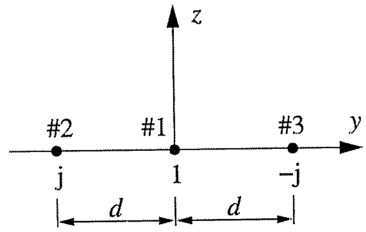
2. The directivity and the effective area are two parameters describing the performance of an antenna in the transmitting and receiving mode respectively. Reciprocity requires that there must be a relation between these parameters.

   a) Derive this relation for a lossless, impedance-matched and polarisation-matched antenna.

   b) Generalize your derived formula by taking into account the antenna radiation efficiency, the reflection efficiency and the polarization loss factor.

   Hint: Consider two antennas separated by a distance $R$ and find the relation between the directivities and effective areas of the antennas. Then, use that the maximum effective area of an isotropic source is $(A_{em})_{isoptopic} = \lambda^2/(4\pi)$

3. A three element array of isotropic sources placed along the $y$ axis has the phase and magnitude relationships shown in the figure below. For a spacing $d = \lambda/4$, derive the array factor of the array.

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4. An infinitesimal horizontal electric dipole of length $l$ is placed parallel to the $x$ axis a height $h$ above an infinite electric ground plane. Find the smallest height $h$ (excluding $h = 0$) that the antenna must be elevated so that the far-field pattern has a null in the $\phi = 30°$ plane at an angle of $\theta = 45°$ from the vertical axis.

   Hint: The far-field components radiated from an infinitesimal electric dipole, of length $l$, placed symmetrically about the origin and directed along the $z$ axis are given by

$$\begin{aligned} \mathbf{E} &= \hat{\theta} E_\theta, \quad \text{where} \quad E_\theta = \mathrm{j}\eta \frac{kI_0 l}{4\pi r} \mathrm{e}^{-\mathrm{j}kr} \sin\theta, \\ \mathbf{H} &= \hat{\phi} H_\phi, \quad \text{where} \quad H_\phi = E_\theta/\eta. \end{aligned}$$

5. Design a constant current circular loop so that its pattern has a null in its far-field pattern at $\theta = 30°$. Find the

   a) smallest possible radius of the loop;

   b) angles where the nulls of the far-field pattern occur.

   Hint: The far-field components radiated by a circular loop of constant current $I_0$ are given by

$$\begin{aligned} E_\phi &= \frac{a\omega\mu I_0}{2r} e^{-\mathrm{j}kr} J_1(ka\sin\theta), \\ H_\theta &= -E_\phi/\eta, \end{aligned}$$

   where $a$ is the radius of the loop and $J_1(z)$ is the Bessel funcion of the first order. Nulls of the Bessel function $[J_1(z_n) = 0]$ occur at $z_1$=0, $z_2$=3.87, $z_3$=7.08, $z_4$=10.17, ...

6. A linear array of isotropic sources is placed along the $z$ axis with the distance $d = \lambda/4$ between the elements. By setting $z = e^{\mathrm{j}\psi}$, where $\psi = kd\cos\theta + \beta$, the array factor of the array can be written as

$$AF = \sum_{n=1}^{N} a_n e^{\mathrm{j}(n-1)\psi} = \sum_{n=1}^{N} a_n z^{n-1} = (z-1)(z-\mathrm{j})(z+1)$$

   a) Determine the progressive phase shift $\beta$ so that the array factor has three nulls.

   b) For your choise of $\beta$, sketch and label the visible region on the unit circle.

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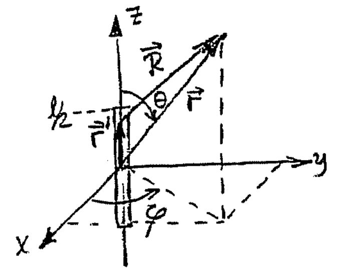



$$\vec{A} = \frac{\mu}{4\pi}\int_C \vec{I}_e(x',y',z')\,\frac{e^{-jkR}}{R}\,dl'$$

Infinitesimal dipole $\Rightarrow$ $\vec{I}_e = \hat{a}_z I_0$

$\vec{R} = \vec{r} - \vec{r}\,'$ $\quad$ $\vec{r}\,' = (x', y', z') = \vec{0}$ for infinitesimal dipole

$R = \vec{r}$ for phase and amplitude

$$\vec{A} = \hat{a}_z \frac{\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} dz' = \hat{a}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}$$

$$\hat{a}_z = \hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta$$

$$\vec{A} = \frac{\mu I_0 l}{4\pi r} e^{-jkr}\left(\hat{a}_r\cos\theta - \hat{a}_\theta \sin\theta\right)$$

$$\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{A} = \frac{1}{\mu}\left\{\hat{a}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\right\} \qquad \vec{A} = (A_r, A_\theta, 0)$$

$$= \frac{1}{\mu}\hat{a}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{\mu I_0 l}{4\pi} e^{-jkr}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\mu I_0 l}{4\pi r} e^{-jkr}\cos\theta\right)\right]$$

$$= \frac{1}{\mu}\hat{a}_\varphi \frac{1}{r}\left[\frac{\mu I_0 l}{4\pi} e^{-jkr}(jk\sin\theta) + \frac{\mu I_0 l}{4\pi r} e^{-jkr}\sin\theta\right]$$

$$= \hat{a}_\varphi \frac{I_0 l}{4\pi r} e^{-jkr}\left[jk\sin\theta + jk\,\frac{\sin\theta}{jkr}\right]$$

$$= \hat{a}_\varphi\, j\frac{kI_0 l}{4\pi r}\sin\theta\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon}\vec{\nabla}\times\vec{H} \qquad \vec{H} = (0, 0, H_\varphi)$$

$$= \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\varphi\sin\theta)\right] + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}(rH_\varphi)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\, j\frac{kI_0 l}{4\pi}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin^2\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right] + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\left(\sin\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[j\frac{kI_0 l}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}\frac{\partial}{\partial\theta}(\sin^2\theta)\right] + \frac{\hat{a}_\theta}{r}\, j\frac{kI_0 l}{4\pi}\sin\theta\left[-\frac{\partial}{\partial r}\left(r\frac{1}{r}\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\, j\frac{kI_0 l}{4\pi}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[1+\frac{1}{jkr}\right]\frac{e^{-jkr}}{r}\,2\sin\theta\cos\theta - \frac{\hat{a}_\theta}{r}\sin\theta\left[-jk - \frac{1}{r} - \frac{1}{jkr^2}\right]e^{-jkr}\right\}$$

$$= \underbrace{\frac{k}{\omega\varepsilon}}_{\eta}\frac{I_0 l}{4\pi}\left\{\hat{a}_r \frac{2}{r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}\cos\theta + \hat{a}_\theta \frac{jk}{r}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}\sin\theta\right\}$$

$$= \hat{a}_r \underbrace{\eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}}_{E_r} + \hat{a}_\theta \underbrace{j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}}_{E_\theta}$$

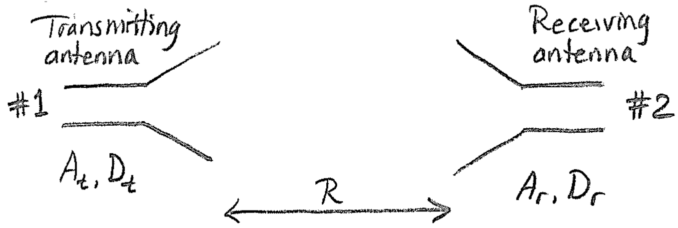
$$H_\phi = j\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$H_r = H_\theta = E_\phi = 0$$

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

2




The power density at a distance $R$ from antenna #1

$$W_t = D_t \frac{P_t}{4\pi R^2} \qquad \left(\text{for isotropic } W_0 = \frac{P_t}{4\pi R^2}\right)$$

Power received by antenna #2

$$P_r = A_r W_t = \frac{P_t D_t A_r}{4\pi R^2}$$

$$\Rightarrow D_t A_r = \frac{P_r}{P_t} 4\pi R^2 \qquad (1)$$

Now, reverse the situation and let antenna #2 be transmitter and antenna #1 receiver. The power density at the distance $R$ from antenna #2 then becomes

$$W_r = D_r \frac{P_t}{4\pi R^2} \qquad \left(\text{for isotropic } W_0 = \frac{P_t}{4\pi R^2}\right)$$

The power received by antenna #1

$$P_r = A_t W_r = \frac{P_t D_r A_t}{4\pi R^2}$$

$$\Rightarrow D_r A_t = \frac{P_r}{P_t} 4\pi R^2 \qquad (2)$$

$(1) = (2) \Rightarrow$

$$D_t A_r = D_r A_t$$

$$\Rightarrow \frac{D_t}{A_t} = \frac{D_r}{A_r} = \text{const} \quad \text{for } \underline{\text{all}} \text{ antennas}$$

Specifically, for an isotropic source we have $D_0 = 1$, $A_{em} = \frac{\lambda^2}{4\pi}$, so

$$\frac{D}{A} = \frac{1}{\lambda^2/(4\pi)}$$

For maximum directivity and maximum effective area

$$\boxed{A_{em} = \frac{\lambda^2}{4\pi} D_0}$$

Taking into account radiation efficiency, impedance-matching, and the PLF:

$$\boxed{A_{em} = e_{cd}(1-|\Gamma|^2)\frac{\lambda^2}{4\pi} D_0 \, |\hat{\rho}_w \cdot \hat{\rho}_a|^2}$$

where $\hat{\rho}_w$ and $\hat{\rho}_a$ are polarization unit vectors of the wave and the antenna, respectively.

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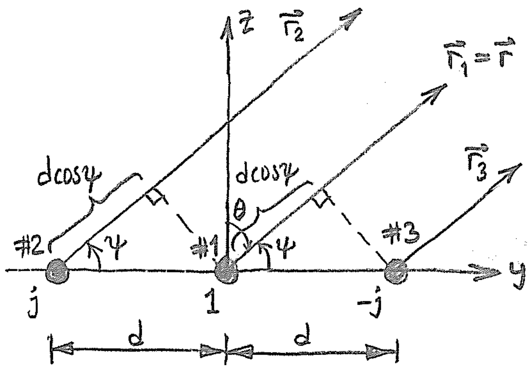



$d = \frac{\lambda}{4}$

$kd = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

$$E = \sum_{n=1}^{N} a_n \frac{e^{-jkr_n}}{r_n} = a_1 \frac{e^{-jkr_1}}{r_1} + a_2 \frac{e^{-jkr_2}}{r_2} + a_3 \frac{e^{-jkr_3}}{r_3}$$

Far-field approximation: $(r_1 \equiv r)$

$$\left.\begin{array}{l} r_2 \approx r + d\cos\psi \\ r_3 \approx r - d\cos\psi \end{array}\right\} \text{for phases}$$

$r_2 \approx r_3$ for amplitudes

$$AF = 1 + je^{-jkd\cos\psi} - je^{jkd\cos\psi}$$

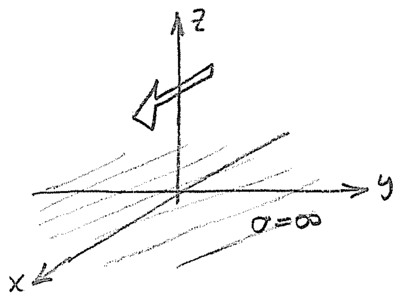
$$= 1 - j\underbrace{\left(e^{jkd\cos\psi} - e^{-jkd\cos\psi}\right)}_{2j\sin(kd\cos\psi)}$$

$$= 1 + 2\sin(kd\cos\psi)$$

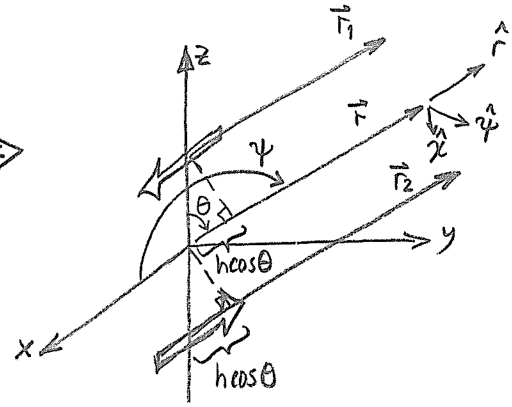
$$\cos\psi = \hat{y}\cdot\hat{r} = \sin\theta\sin\phi$$

$$AF = 1 + 2\sin\left(\frac{\pi}{2}\sin\theta\sin\phi\right)$$

4.




$\Rightarrow$




Introduce a new spherical coordinate system $(r, \psi, \chi)$

$$E_\psi = E_\psi^d + E_\psi^r \quad \text{where}$$

$$E_\psi^d = j\eta \frac{kI_0 l}{4\pi r_1} e^{-jkr_1} \sin\psi$$

$$E_\psi^r = -j\eta \frac{kI_0 l}{4\pi r_2} e^{-jkr_2} \sin\psi$$

$$\vec{H} = \frac{1}{\eta} \hat{r} \times \vec{E}$$

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{x}\cdot\hat{r}|^2}$$
$$= \sqrt{1-\sin^2\theta\cos^2\phi}$$

Far-field approximation

$$\left.\begin{aligned} r_1 &\approx r - h\cos\theta \\ r_2 &\approx r + h\cos\theta \end{aligned}\right\} \text{for phases}$$

$r_1 \approx r_2 \approx r$ for amplitudes

$$E_\psi = j\eta \frac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi \underbrace{\left[e^{jkh\cos\theta} - e^{-jkh\cos\theta}\right]}_{2j\sin(kh\cos\theta)}$$

$$= j\eta \frac{kI_0 l}{4\pi r} e^{-jkr} \sqrt{1-\sin^2\theta\cos^2\phi}\left[2j\sin(kh\cos\theta)\right]$$

Null for $\theta = 45^\circ$, $\phi = 30^\circ$

$$\Rightarrow \sqrt{1-\sin^2\theta\cos^2\phi}\, \sin(kh\cos\theta)\Big|_{\substack{\theta=45^\circ \\ \phi=30^\circ}} = 0$$

$$\Rightarrow \sqrt{1-\sin^2 45^\circ \cos^2 30^\circ} \neq 0 \quad \text{and} \quad \sin(kh\cos 45^\circ) = 0$$

$$\sin\left(\frac{2\pi}{\lambda} h \frac{1}{\sqrt{2}}\right) = \sin\left(\frac{\sqrt{2}\pi h}{\lambda}\right) = 0$$

$$\frac{\sqrt{2}\pi h}{\lambda} = n\pi \qquad n = 0, 1, \ldots$$

$n = 1$ gives the lowest height $h = \frac{\lambda}{\sqrt{2}}$

5. The nulls of the Bessel function $J_1$ are $0;\ 3.87;\ 7.08;\ 10.17;\ \ldots$

The first null gives

$$ka\sin\theta = 0 \quad \Rightarrow \sin\theta = 0 \Rightarrow \begin{array}{l}\theta_1 = 0^\circ \\ \theta_2 = 180^\circ\end{array}$$

a) The second null gives the smallest radius of the loop when $\theta_3 = 30^\circ$

$$ka\sin 30^\circ = 3{,}87$$

$$\Rightarrow a = \frac{3{,}87}{k\underbrace{\sin 30^\circ}_{1/2}} = \frac{3{,}87\lambda}{2\pi\cdot 1/2} = \frac{3{,}87\lambda}{\pi} \approx \underline{\underline{1{,}23\lambda}}$$

b) The null $\theta_3 = 30^\circ$ gives also $\theta_4 = 180^\circ - 30^\circ = 150^\circ$

The 3rd null gives

$$ka\sin\theta = 7.08$$

$$\Rightarrow \sin\theta = \frac{7.08\lambda}{2\pi a} = \frac{7{,}08}{2\pi\cdot 1{,}23} \Rightarrow \begin{array}{l}\theta_5 = 66{,}2^\circ \\ \theta_6 = 113{,}8^\circ\end{array}$$

The 4th null gives

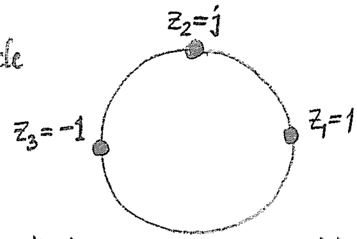
$$\sin\theta = \frac{10.17}{2\pi\cdot 1{,}23} = 1{,}31 > 0 \Rightarrow \text{No more nulls}$$

Answer:
a) The radius: $a = 1{,}23\lambda$
b) Nulls at $\theta = 0^\circ$
$30^\circ$
$66{,}2^\circ$
$113{,}8^\circ$
$150^\circ$
$180^\circ$

6. a) The AF has three nulls $z_1 = 1$, $z_2 = j$, $z_3 = -1$ only if all these nulls are within the visible region of the unit circle.

$z = e^{j\psi}$

$|z| = 1 \Rightarrow$ unit circle



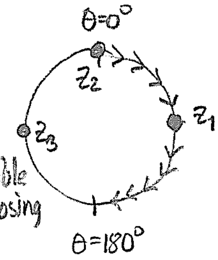

We must choose $\beta$ so that $z_1, z_2, z_3$ are within the visible region

$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta + \beta = \frac{\pi}{2}\cos\theta + \beta$$

$$\theta = 0^\circ \Rightarrow \psi = \frac{\pi}{2}\cos 0^\circ + \beta = \frac{\pi}{2} + \beta$$

$$\theta = 180^\circ \Rightarrow \psi = \frac{\pi}{2}\cos 180^\circ + \beta = -\frac{\pi}{2} + \beta$$

For $\beta = 0$: We must rotate the visible region by choosing $\beta \neq 0$




If we choose $\beta = \frac{\pi}{2}$

$$\psi(\theta = 0^\circ) = \pi$$

$$\psi(\theta = 180^\circ) = 0$$

$\Rightarrow z_1, z_2, z_3$ are all within the visible region

b)