Exam in *Antenna Theory*
Time: 20 Mars 2006, at 09.00–14.00.
Location: Gimogatan 4, Sal 2

You may bring: Laboratory reports, pocket calculator, English dictionary,
Råde-Westergren: "Beta", Nordling-Österman: "Physics Handbook", or comparable handbooks.

Six problems, maximum five points each, for a total maximum of 30 points.

1. Consider an infinitesimal electric dipole of length l placed symmetically at the origin. Assume that the current in the dipole is constant and given by $\mathbf{I}_e(z') = \hat{z}I_0$. Find the electic and magnetic field components radiated by the dipole in *all* space.
   Hints: The vector potential $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{\mu}{4\pi}\int \mathbf{I}_e(x', y', z')\frac{e^{-jkR}}{R}\mathrm{d}l'$$

   where R is the distance from any point on the source to the observation point. The magnetic and electric fields are given by

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu}\nabla \times \mathbf{A} \\ \mathbf{E} &= -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A}) = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} \end{aligned}$$

2. The normalized far-zone field pattern of an antenna is given by

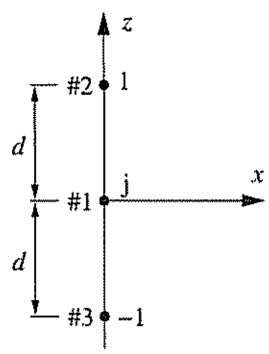
$$E = \begin{cases} \sqrt{\sin^2\theta\,\sin\phi} & 0 \le \theta \le \pi,\ 0 \le \phi \le \pi \\ 0 & \text{elsewhere} \end{cases}$$

   Find the directivity using

   a) the exact expression;
   b) Kraus' approximative formula.

3. A three element array of isotropic sources has the phase and magnitude relationships shown in the Figure below. The spacing between the elements is $d = \lambda/2$.

   a) Find the array factor.
   b) Find all the nulls.



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4. An infinitesimal vertical linear magnetic dipole of constant magnetic current I_{m} and length l is placed a distance h above an infinite perfectly conducting electric ground plane.

   a) Find the far-zone **E**- and **H**-field components radiated by the dipole in the presence of the ground plane.
   b) Specify the angular region where the radiated fields are valid.

   Hint: The far-field components radiated from an infinitesimal electric dipole placed along the z axis are given by

$$\begin{aligned} \mathbf{E} &= \hat{\theta}\, E_\theta, \quad \text{where} \quad E_\theta = \mathrm{j}\eta \frac{k I_0 l}{4\pi r} \mathrm{e}^{-\mathrm{j}kr} \sin\theta, \\ \mathbf{H} &= \hat{\phi}\, H_\phi, \quad \text{where} \quad H_\phi = E_\theta/\eta. \end{aligned}$$

5. Design a constant current circular loop so that its far-field pattern has exactly two nulls above the plane of the loop at the angles $\theta = 0^\circ$ and $\theta = 60^\circ$, where θ is meaured from the axis of the loop.

   a) Find the radius a of the loop.
   b) Find the angles of all the nulls.

   Hint: The far-field components radiated by a circular loop of radius a and constant current I_0 are given by

$$\begin{aligned} E_\phi &= \frac{a\omega\mu I_0}{2r} \mathrm{e}^{-\mathrm{j}kr} J_1(ka\sin\theta), \\ H_\theta &= -E_\phi/\eta, \end{aligned}$$

   where $J_1(z)$ is the Bessel function of the first order. Nulls of the Bessel function $[J_1(z_n) = 0]$ occur at $z_1 = 0$, $z_2 = 3.87, \ldots$

6. Design a linear array of isotropic elements placed along the z axis such that the nulls of the array factor occur at $\theta = 0^\circ$, $\theta = 90^\circ$ and $\theta = 180^\circ$. Assume that the elements are spaced a distance $d = 3\lambda/4$ apart and that $\beta = 0^\circ$.

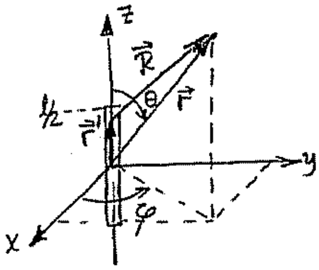
   a) Sketch and label the visible region on the unit circle.
   b) Find the required number of elements.
   c) Determine the excitation coefficients of the elements.

   Hint: The array factor of an N-element linear array is given by

$$AF = \sum_{n=1}^{N} a_n \mathrm{e}^{\mathrm{j}(n-1)\psi},$$

   where $\psi = kd\cos\theta + \beta$. Use the representation $z = \mathrm{e}^{\mathrm{j}\psi}$.

1



$$\vec{A} = \frac{\mu}{4\pi}\int_C \vec{I}_e(x',y',z')\,\frac{e^{-jkR}}{R}\,dl'$$

Infinitesimal dipole $\Rightarrow$ $\vec{I}_e = \hat{a}_z I_0$

$\vec{R} = \vec{r} - \vec{r}\,'$ $\qquad \vec{r}\,' = (x', y', z') = \vec{0}$ for infinitesimal dipole

$R = \vec{r}$ for phase and amplitude

$$\vec{A} = \hat{a}_z \frac{\mu I_0}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} dz' = \hat{a}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}$$

$$\hat{a}_z = \hat{a}_r \cos\theta - \hat{a}_\theta \sin\theta$$

$$\vec{A} = \frac{\mu I_0 l}{4\pi r} e^{-jkr}\left(\hat{a}_r\cos\theta - \hat{a}_\theta\sin\theta\right)$$

$$\vec{H} = \frac{1}{\mu}\vec{\nabla}\times\vec{A} = \frac{1}{\mu}\left\{ \hat{a}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\right\} \qquad \vec{A} = (A_r, A_\theta, 0)$$

$$= \frac{1}{\mu}\hat{a}_\varphi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{\mu I_0 l}{4\pi} e^{-jkr}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{\mu I_0 l}{4\pi r} e^{-jkr}\cos\theta\right)\right]$$

$$= \frac{1}{\mu}\hat{a}_\varphi \frac{1}{r}\left[\frac{\mu I_0 l}{4\pi} e^{-jkr}(jk\sin\theta) + \frac{\mu I_0 l}{4\pi r} e^{-jkr}\sin\theta\right]$$

$$= \hat{a}_\varphi \frac{I_0 l}{4\pi r} e^{-jkr}\left[jk\sin\theta + jk\,\frac{\sin\theta}{jkr}\right]$$

$$= \hat{a}_\varphi\, j\frac{kI_0 l}{4\pi r}\sin\theta\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon}\vec{\nabla}\times\vec{H} \qquad \vec{H} = (0, 0, H_\varphi)$$

$$= \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\varphi\sin\theta)\right] + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}(rH_\varphi)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\, j\frac{kI_0 l}{4\pi}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin^2\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right] + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\left(\sin\theta\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[j\frac{kI_0 l}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}\frac{\partial}{\partial\theta}(\sin^2\theta)\right] + \frac{\hat{a}_\theta}{r}\, j\frac{kI_0 l}{4\pi}\sin\theta\left[-\frac{\partial}{\partial r}\left(r\frac{1}{r}\left[1+\frac{1}{jkr}\right]e^{-jkr}\right)\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon}\, j\frac{kI_0 l}{4\pi}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[1+\frac{1}{jkr}\right]\frac{e^{-jkr}}{r}\,2\sin\theta\cos\theta - \frac{\hat{a}_\theta}{r}\sin\theta\left[-jk - \frac{1}{r} - \frac{1}{jkr^2}\right]e^{-jkr}\right\}$$

$$= \underbrace{\frac{k}{\omega\varepsilon}}_{\eta}\,\frac{I_0 l}{4\pi}\left\{\hat{a}_r\,\frac{2}{r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}\cos\theta + \hat{a}_\theta\,\frac{jk}{r}\left[1+\frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}\sin\theta\right\}$$

$$= \hat{a}_r \underbrace{\eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}}_{E_r} + \hat{a}_\theta \underbrace{j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}}_{E_\theta}$$

$$H_\varphi = j\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$H_r = H_\theta = E_\varphi = 0$$

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

2.

$$E = \begin{cases} \sqrt{\sin^2\theta \sin\phi} & 0 \leq \theta \leq \pi \\ & 0 \leq \phi \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

$$U(\theta,\phi) = \frac{r^2}{2\eta}|\bar{E}|^2 \sim \frac{r^2}{2\eta}\sin^2\theta\sin\phi$$

$$U_{max} = U\left(\theta=\frac{\pi}{2}, \phi=\frac{\pi}{2}\right) = \frac{r^2}{2\eta}$$

$$U(\theta,\phi) = U_{max}\sin^2\theta\sin\phi$$

a) $D_o = \dfrac{4\pi U_{max}}{P_{rad}}$

$$P_{rad} = \oint_{\Omega} U(\theta,\phi)d\Omega = U_{max}\int_0^{\pi}\int_0^{\pi}\sin^2\theta\sin\phi\sin\theta\, d\theta d\phi$$

$$= U_{max}\int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta\int_0^{\pi}\sin\phi\, d\phi = U_{max}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}\left[-\cos\phi\right]_0^{\pi}$$

$$= U_{max}\left(1-\frac{1}{3}+1-\frac{1}{3}\right)(1+1) = \frac{8}{3}U_{max}$$

$$D_o = \frac{4\pi U_{max}}{8U_{max}/3} = \frac{3\pi}{2}$$

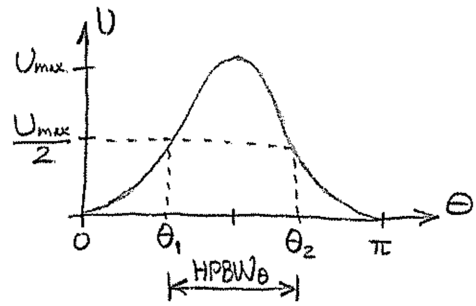
b) For $\phi = 90° = \frac{\pi}{2}$, find $HPBW_\theta$

$$\frac{U_{max}}{2} = U_{max}\sin^2\theta$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta_1 = \frac{\pi}{4}$$

$$\theta_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$HPBW_\theta = \theta_2 - \theta_1 = \frac{\pi}{2}$$



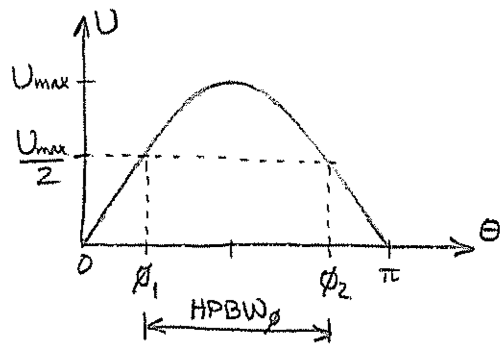
For $\theta = 90° = \frac{\pi}{2}$, find $HPBW_\phi$

$$\frac{U_{max}}{2} = U_{max}\sin\phi$$

$$\Rightarrow \sin\phi = \frac{1}{2} \Rightarrow \phi_1 = \frac{\pi}{6}$$

$$\phi_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$HPBW_\phi = \phi_2 - \phi_1 = \frac{2\pi}{3}$$



Kraus: $D_o = \dfrac{4\pi}{\Theta_{1r}\Theta_{2r}}$ where $\Theta_{1r} = HPBW_\theta$, $\Theta_{2r} = HPBW_\phi$

$$D_o = \frac{4\pi}{\frac{\pi}{2}\cdot\frac{2\pi}{3}} = \frac{12}{\pi}$$

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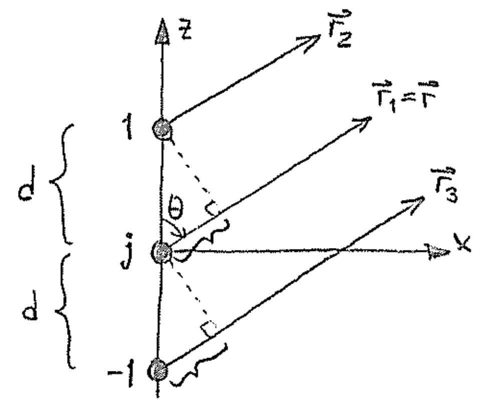


$d = \lambda/2$

$kd = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$

a) Far-field approximation:

$$\left.\begin{aligned} r_1 &= r \\ r_2 &\approx r - d\cos\theta \\ r_3 &\approx r + d\cos\theta \end{aligned}\right\} \text{for phases}$$

$$r_2 \approx r_3 \approx r_1 = r \qquad \text{for amplitudes}$$

$$E \sim \sum_{n=1}^{N} a_n \frac{e^{-jkr_n}}{r_n}$$

$$AF = j + e^{jkd\cos\theta} - e^{-jkd\cos\theta} = j + 2j\sin(kd\cos\theta)$$

$$= j\left[1 + 2\sin(kd\cos\theta)\right] = j\left[1 + 2\sin(\pi\cos\theta)\right]$$

b) Nulls when $AF = 0 \Rightarrow 1 + 2\sin(\pi\cos\theta) = 0$

$\Rightarrow \sin(\pi\cos\theta) = -\frac{1}{2}$

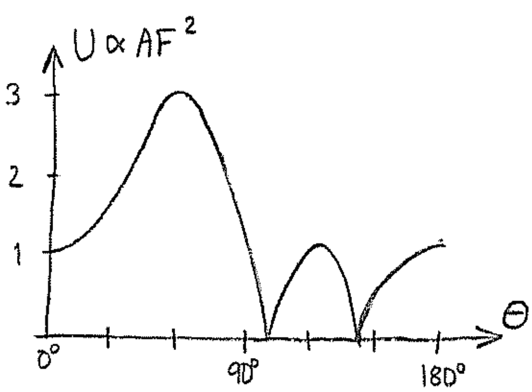
$\Rightarrow \pi\cos\theta = -\frac{\pi}{6} + n\cdot 2\pi$ or $\pi\cos\theta = \pi - \left(-\frac{\pi}{6}\right) + n\cdot 2\pi \quad n = 0, \pm 1, \ldots$

$= \frac{7\pi}{6} + n\cdot 2\pi$

$\Rightarrow \cos\theta = -\frac{1}{6} + n\cdot 2,\ n = 0$ $\qquad \cos\theta = \frac{7}{6} + \underbrace{n}_{-1}\cdot 2 = -\frac{5}{6}$

$\Rightarrow \theta = \overset{+}{(-)}\, 99.6°$ $\qquad \theta = \overset{+}{(-)}\, 146.4°$

Nulls at $\theta = 99.6°$ and $146.4°$



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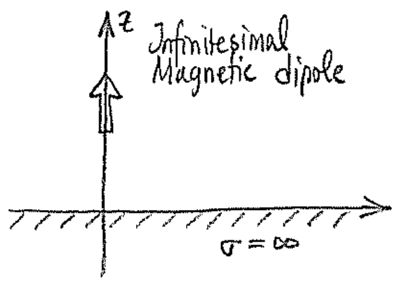
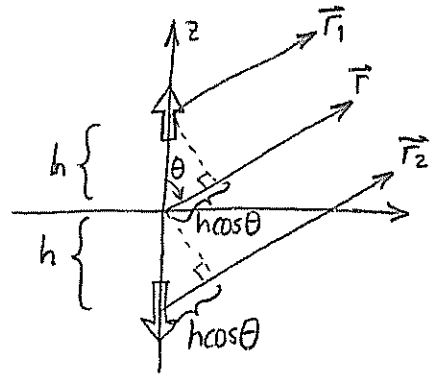


$\Longleftrightarrow$



Fields from a vertical infinitesimal electric dipole

$$\begin{cases} E_\theta = j\eta \dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\theta \\ H_\phi = E_\theta/\eta \end{cases}$$

Duality $\Rightarrow$ fields from an infinitesimal magnetic dipole $\Leftrightarrow$ $\begin{cases} \vec{E} \rightarrow \vec{H}^{(m)} \\ \vec{H} \rightarrow -\vec{E}^{(m)} \\ \vec{I}_e \rightarrow \vec{I}_m \\ \eta \rightarrow 1/\eta \end{cases}$

$$E_\phi = -j \frac{kI_m l}{4\pi r} e^{-jkr} \sin\theta$$

Take into account the mirror source

$$E_\phi = E_\phi^{(1)} + R E_\phi^{(2)}$$

where

$$E_\phi^{(1)} = -j \frac{kI_m l}{4\pi r_1} e^{-jkr_1} \sin\theta$$

$$E_\phi^{(2)} = -j \frac{kI_m l}{4\pi r_2} e^{-jkr_2} \sin\theta$$

$$R = -1$$

Far-field approximation

$$\left.\begin{aligned} r_1 &\approx r - h\cos\theta \\ r_2 &\approx r + h\cos\theta \end{aligned}\right\} \text{ for phases}$$

$$r_1 \approx r_2 \approx r \qquad \text{for amplitudes}$$

$$E_\phi = -j \frac{kI_m l}{4\pi r} e^{-jkr} \sin\theta \underbrace{\left(e^{jkh\cos\theta} - e^{-jkh\cos\theta}\right)}_{2j\sin(kh\cos\theta)}$$

$$= \frac{kI_m l}{2\pi r} e^{-jkr} \sin\theta \sin(kh\cos\theta)$$

$$\vec{H} = \frac{1}{\eta}\hat{k} \times \vec{E} = \frac{1}{\eta}\hat{r} \times \vec{E} = \frac{1}{\eta}\hat{r} \times \hat{\phi} E_\phi = -\frac{E_\phi}{\eta}\hat{\theta}$$

$$H_\theta = -\frac{kI_m l}{2\pi\eta r} e^{-jkr} \sin\theta \sin(kh\cos\theta)$$

5 Null $\Rightarrow$ $J_1(ka\sin\theta)=0$ Nulls at $\theta = 0°, 60°$

$\Rightarrow$ $ka\sin\theta = 0;\ 3{,}87;\ \ldots$

a) First null of J_1: $ka\sin\theta = 0$

$$\Rightarrow \theta_1 = 0°, \quad \theta_2 = 180° - 0° = 180°$$

2nd null of J_1: $ka\sin 60° = 3{,}87$

$$\Rightarrow a = \frac{3{,}87}{k\sin 60°} = \frac{3{,}87\lambda}{2\pi\cdot\sqrt{3}/2} = \underline{\underline{0{,}711\lambda}}$$

In general $\sin\theta_3 = \frac{3{,}87}{ka} = \frac{3{,}87\lambda}{2\pi a}$

$$\Rightarrow \theta_3 = \arcsin\left(\frac{3{,}87\lambda}{2\pi a}\right) = 60°$$

and

$$\theta_4 = 180° - \theta_3 = 180° - 60° = 120°$$

In total four nulls at $\theta = 0°, 60°, 120°$ and $180°$

6 Nulls of AF at $\theta = 0^\circ, 90^\circ, 180^\circ$

$d = 3\lambda/4$

$\beta = 0$

a) $AF = \sum_{n=1}^{N} a_n e^{j(n-1)\psi}$ Set $z = e^{j\psi}$

$$AF = \sum_{n=1}^{N} a_n z^{n-1} = a_1 + a_2 z + a_3 z^2 + \dots + a_N z^{N-1}$$

$$= (z - z_1)(z - z_2) \cdot \dots \cdot (z - z_{N-1})$$ where z_n are nulls of the AF

$$\psi = kd\cos\theta + \beta = \frac{2\pi}{\lambda}\frac{3\lambda}{4}\cos\theta + 0 = \frac{3\pi}{2}\cos\theta$$

$\theta = 0^\circ \Rightarrow \psi(\theta = 0^\circ) = \frac{3\pi}{2} \quad \left(= -\frac{\pi}{2}\right)$

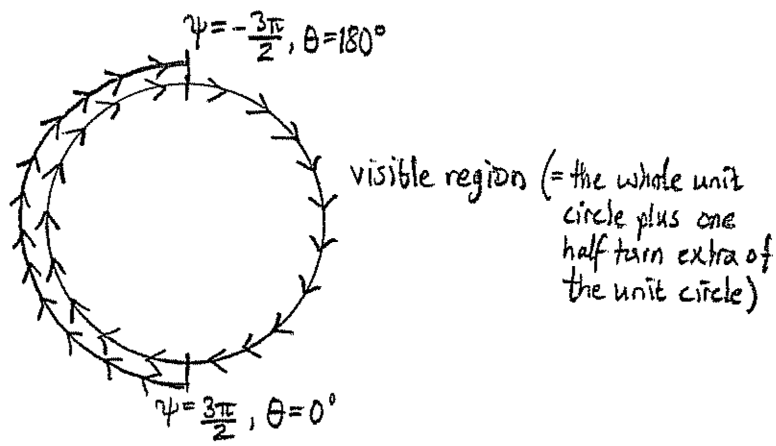
$\theta = 90^\circ \Rightarrow \psi(\theta = 90^\circ) = 0$

$\theta = 180^\circ \Rightarrow \psi(\theta = 180^\circ) = -\frac{3\pi}{2} \quad \left(= \frac{\pi}{2}\right)$

When θ varies from 0° to 180°, ψ varies from $\frac{3\pi}{2}$, trough 0 and ends at $-\frac{3\pi}{2}$. The visible region is $\frac{3\pi}{2} \geq \psi \geq -\frac{3\pi}{2}$ for $0 \leq \theta \leq 180^\circ$

$z = e^{j\psi}$

$|z| = 1 \Rightarrow$ unit circle



b) Nulls:

$\theta_1 = 0 \Rightarrow \psi_1 = \frac{3\pi}{2}\cos 0 = \frac{3\pi}{2} \quad \Rightarrow \quad z_1 = e^{j\psi_1} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}} = -j$

$\theta_2 = \pi/2 \Rightarrow \psi_2 = \frac{3\pi}{2}\cos\frac{\pi}{2} = 0 \quad \Rightarrow \quad z_2 = e^{j\psi_2} = e^{j0} = 1$

$\theta_3 = \pi \Rightarrow \psi_3 = \frac{3\pi}{2}\cos\pi = -\frac{3\pi}{2} \quad \Rightarrow \quad z_3 = e^{j\psi_3} = e^{-j\frac{3\pi}{2}} = e^{j\frac{\pi}{2}} = j$

$AF = (z - z_1)(z - z_2)(z - z_3) = a_1 + a_2 z + a_3 z^2 + z^3$ = Eq with 3 roots (nulls) $\Rightarrow$ <u>4 elements</u> required

c) $AF = (z - z_1)(z - z_2)(z - z_3) = \dots =$

$$= z^3 - z^2(z_1 + z_2 + z_3) + z(z_1 z_2 + z_2 z_3 + z_3 z_1) - z_1 z_2 z_3$$

$a_1 = -z_1 z_2 z_3 = -(-j) \cdot 1 \cdot j = -1$

$a_2 = z_1 z_2 + z_2 z_3 + z_3 z_1 = -j \cdot 1 + 1 \cdot j + j(-j) = -j + j + 1 = 1$

$a_3 = -(z_1 + z_2 + z_3) = -(-j + 1 + j) = -1$

$a_4 = 1$